ISBN 86–80593–29–X

m-M CALCULUS

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(Second revised edition)

Matematički institut
Beograd
1997

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CONTENTS

0. INTRODUCTION

This is the second revised edition of the m-M Calculus (see [7]).

In this paper we consider the so-called m-M functions, i.e. functions of the form

$f : D \to \mathbb{R}$ ($D = [a_1, b_1] \times ... \times [a_n, b_n]$, where $n > 0$ is any element of $\mathbb{N}$; and $a_i, b_i \in \mathbb{R}$)

subjected to the following supposition:

For each n-dimensional segment $\Delta = [\alpha_1, \beta_1] \times ... \times [\alpha_n, \beta_n] \subset D$ a pair of real numbers, denoted by $m(f)(\Delta)$, $M(f)(\Delta)$, satisfying the conditions

(0.1) $m(f)(\Delta) \leq f(X) \leq M(f)(\Delta)$ (for all $\Delta \subset D$, $X \in \Delta$)

(0.2) $\lim_{diam\Delta \to 0}(M(f)(\Delta) - m(f)(\Delta)) = 0$ (where $diam\Delta := (\sum(\beta_i - \alpha_i)^2)^{1/2}$)

is effectively given.

Such an ordered pair $\langle m(f), M(f)\rangle$ of mappings $m(f)$,$M(f)$ (both mapping the set of all $\Delta \subset D$ into $\mathbb{R}$) is called an **m-M pair** of the function f. We also say that $m(f)$, $M(f)$ are **generalized minimum** and **maximum** for f respectively. With only a few exceptions, all elementary functions are m-M functions (Lemma 1.4).

The conditions (0.1) and (0.2) are taken as axioms of the so-called **m-M calculus** (or the **Calculus of generalized minimum and maximum**).

A logical analysis of these axioms is given here and, in addition to the other results, a series of equivalences is proved which enable us to express some relationships for m-M functions by means of the corresponding relationships for their m-M pairs (Formulas (4.8), (4.9), (4.10)).

There are many various applications of the m-M calculus, such as

- *Solving systems of inequalities, systems of equations (Section 2)*
- *Finding n-dimensional integrals (Section 3)*
- *Solving any problem expressed by a positive $\leq$-formula (Section 5). Among others*
    - *Problem of constrained optimization (Problem 5.2, Problem 5.3)*
    - *Problem of unconstrained optimization (Problem 5.1)*
    - *min-max problems (Problem 5.4)*
    - *Problems from Interval Mathematics (Problem 5.5)*
- *Finding functions satisfying a given m-M condition (e.g. functional condition, or difference equation, or differential equation)*

'As it is well known, by the usual methods of Numerical analysis, assuming certain convergence condiitons, we aproximately determine, step-by-step, one solution of the given problem (see [1]-[5]). However, applying the methods of m-M calculus we approximately determine all solutions of the given problem, and we assume almost nothing about the convergence. The solutions are, as a rule, sought in a prescribed n-dimensional segment D. If the given problem, e.g. a system oof some equations, has no solutions in D, then applying the method of m-M calculus this can be established at a certain finite step k. The basic methodological idea of the m-M calculus is:

> It gives a sufficient condition Cond(Δ) which ensures that an n-dimensional segment Δ does not contain any solutions of the considered problem P. Applying this criterion, we reject from the original n-segment D those "pieces" which do not contain solutions, so that in the limiting case the remaining "pieces" form the set S of all solutions of the problem P (if indeed there is a solution of P).

Compairing with [7] this version of the m-M Calculus additionally contains

> A linear procedure LS by which one can find a local minimum or saddle point of the function from Problem 5.1 (see Remark 5.1).
>
> On the disjunctive-optimization problem (see Remark 5.2).
>
> 7. Appendix, containing a new, simple proof of the Theorem 4.3, which is the key theorem of the m-M Calculus.

The author is indebted to M. Agović who designed several programs concerning the m-M Calculus and to Professor M. Rašković for several valuable comments and suggestions.

1. m-M ALGEBRA

In this section we state how for a given function f defined by some elementary term $f(x_1, ..., x_n)$ (see (1.7) and (1.11)) one can, in a finite number of steps, find an m-M pair. Also we study some general properties of m-M pairs.

1. Throughout this section we shall denote by

$$D = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$$

a fixed n-dimensional segment. The functions we deal with are mainly some m-M functions $f : D \to \mathbb{R}$.

As the first fact notice that from axioms (0.1), (0.2) follows that the function f must be continuous; i.e. if $f : D \to \mathbb{R}$ is an m-M function then f must be continuous in D. It is easy to see that in some sense the opposite assertion is also true. Namely, if $f : D \to \mathbb{R}$ is a given continuous function then one of its m-M pairs may be defined by

$$(1.1) \qquad m(f)(\Delta) = \min_{X\in\Delta} f(X), \quad M(f)(\Delta) = \max_{X\in\Delta} f(X).$$

Notice that this m-M pair satisfies the following implication

$$(1.2) \quad \Delta' \subseteq \Delta'' \Rightarrow m(f)(\Delta'') \leq m(f)(\Delta'), M(f)(\Delta') \leq m(f)(\Delta'') \quad (\Delta'' \subseteq D)$$

Generally, any m-M pair having this property will be called a monotone m-M pair. Here and throughout the section we denote by $\Delta = [\alpha_1, \beta_1] \times ... \times [\alpha_n, \beta_n]$ any subsegment of D. Further, the set of all such Δ's will be denoted by $Int(D)$. We point out that formula (1.1) can be used in case of monotone functions. Namely, we have the following lemma.

Lemma 1.1. *Let $f : D \to \mathbb{R}$ be a continuous function, monotone in each of its arguments. One m-M pair of f, the so-called ideal m-M pair, can be defined by the following equalities*

$$m(f)(\Delta) = \min\{f(V_1), ..., f(V_{2^n})\}, \qquad M(f)(\Delta) = \max\{f(V_1), ..., f(V_{2^n})\}$$

where V_1, ..., V_{2^n} are all the vertices of Δ. This m-M pair is monotone.

The proof follows by (1.1) and the fact that f, when $X \in \Delta$, must acheive its minimum(maximum) at some vertex of Δ. From Lemma 1.1 we obtain the following corollary

(1.3) If $f : [a_1, b_1] \to \mathbb{R}$ is a continuous monotone function then one of its m-M pairs is determined by the equality

(i) $m(f)(\Delta) = f(\alpha_1)$, $M(f)(\Delta) = f(\beta_1)$ if f is nondecreasing

or

(ii) $m(f)(\Delta) = f(\beta_1)$, $M(f)(\Delta) = f(\alpha_1)$ if f is nonincreasing

As another corollary we list the following table in which f denotes the function defined by the given expression and $\langle m(f)(\Delta), M(f)(\Delta)\rangle$ is one m-M pair of f.

Table 1.1

Function f	$m(f)(\Delta)$	$M(f)(\Delta)$	Under condition
C	C	C	C is a constant
x_1	α_1	β_1	
x_1+x_2	$\alpha_1+\alpha_2$	$\beta_1+\beta_2$	
$-x_1$	$-\beta_1$	$-\alpha_1$	
$1/x_1$	$1/\beta_1$	$1/\alpha_1$	$\alpha_1>0$
$x_1\cdot x_2$	$\min(\alpha_1\alpha_2,\alpha_1\beta_2,\alpha_2\beta_1,\alpha_2\beta_2)$	$\max(\alpha_1\alpha_2,\alpha_1\beta_2,\alpha_2\beta_1,\alpha_2\beta_2)$	
$\min(x_1,x_2)$	$\min(\alpha_1,\alpha_2)$	$\min(\beta_1,\beta_2)$	
$\max(x_1,x_2)$	$\max(\alpha_1,\alpha_2)$	$\max(\beta_1,\beta_2)$	

In connection with formulas (1.1) we also emphasize the following fact. If f is any m-M function then for each its m-M pair the following inequalities

(1.4) $$m(f)(\Delta)\le\min_{X\in\Delta}f(X),\quad M(f)(\Delta)\ge\max_{X\in\Delta}f(X)$$

hold. More precisely said we have the following lemma.

Lemma 1.2. *Let $f:D\to\mathbb{R}$ be a given continuous function and let $\varepsilon_1,\varepsilon_2:Int(D)\to\mathbb{R}$ be any given nonnegative functions with the property*

$$\lim_{diam\Delta\to0}\varepsilon_1(\Delta)=\lim_{diam\Delta\to0}\varepsilon_2(\Delta)=0$$

Then one m-M pair of the function f can be defined by the following equalities

(*) $$m(f)(\Delta)=\min_{X\in\Delta}f(X)-\epsilon_1(\Delta),\quad M(f)(\Delta)=\max_{X\in\Delta}f(X)+\epsilon_2(\Delta)$$

Moreover, each m-M pair of f can be represented in the form (*)

Now let $f:D\to\mathbb{R}$ be any m-M function. In connection with it we introduce two functions (mf), (Mf) of $2n$ arguments. These functions are only partialy defined. Namely, if $\Delta=[\alpha_1,\beta_1]\times...\times[\alpha_n,\beta_n]$ is any subsegment of D then, by definition, we have:

(1.5) $$(mf)(\alpha_1,\beta_1,...,\alpha_n,\beta_n):=m(f)(\Delta),\quad (Mf)(\alpha_1,\beta_1,...,\alpha_n,\beta_n):=M(f)(\Delta)$$

In other words if we like to consider $m(f)(\Delta)$ and $M(f)(\Delta)$ as functions of α_1, β_1, ..., α_n, β_n then we can do it using the functions (mf), (Mf).

Generally speaking, regarding the notion of monotony we have the following fact

(1.6) *If $\langle m(f),M(f)\rangle$ is a monotone m-M pair then the functions (mf), (Mf) are also monotone in each of their arguments.*

Lemma 1.3. *Let the functions $f,h_1,...,h_k:D\to\mathbb{R}$ and*

$$G:[A_1,B_1]\times...\times[A_k,B_k]\to\mathbb{R}\quad (A_i,B_i \text{ are given reals})$$

satisfy the following equality

$$f(X)=G(h_1(X),...,h_k(X))\quad (\text{for all } X\in D)$$

Suppose also that for all segments $\Delta\subseteq D$ the inequalities

(*) $$A_i\le m(h_i)(\Delta),M(h_i)(\Delta)\le B_i\quad (i=1,...,k)$$

are fulfilled. Then one m-M pair of the function f is defined by the equalities

(**) $$\begin{aligned}m(f)(\Delta)&=(mG)(m(h_1)(\Delta),M(h_1)(\Delta),...,m(h_k)(\Delta),M(h_k)(\Delta))\\ M(f)(\Delta)&=(MG)(m(h_1)(\Delta),M(h_1)(\Delta),...,m(h_k)(\Delta),M(h_k)(\Delta))\end{aligned}$$

providing that all m-M pairs of the functions h_1, ..., h_k, G standing on the right hand side of these equalities are known.

Proof. Let $\Delta\subseteq D$ and $X\in\Delta$. Denote the product

$$[m(h_1)(\Delta),M(h_1)(\Delta)]\times...\times[m(h_k)(\Delta),M(h_k)(\Delta)]$$

by Δ'. Then we have

$$\begin{aligned}f(X)&=G(h_1(X),...,h_k(X))\\ &\le M(G)(\Delta')=(MG)(m(h_1)(\Delta),M(h_1)(\Delta),...,m(h_k)(\Delta),M(h_k)(\Delta))\end{aligned}$$

Similarly the inequality

$$f(X)\ge(mG)(m(h_1)(\Delta),M(h_1)(\Delta),...,m(h_k)(\Delta),M(h_k)(\Delta))$$

can be proved. Thus axiom (0.1) is satisfied. Further:

$$\begin{aligned}&\lim_{diam\Delta\to0}[M(G)(m(h_1)(\Delta),M(h_1)(\Delta),...,m(h_k)(\Delta),M(h_k)(\Delta))\\ &\qquad -m(G)(m(h_1)(\Delta),M(h_1)(\Delta),...,m(h_k)(\Delta),M(h_k)(\Delta))]\\ &=\lim_{diam\Delta\to0}(M(G)(\Delta')-m(G)(\Delta'))=0\quad (\text{For } \lim\Delta'=0)\end{aligned}$$

Consequently the axiom (0.2) is satisfied, which completes the proof.

Remark 1.1. *Note that Lemma 1.3 is compatible with the monotony property. Namely, if m-M pairs of the functions h_1, ..., h_k, G are monotone then the m-M pair of the function f determined by the lemma is monotone too.*

Indeed, let $\Delta''\subseteq\Delta'\subseteq\Delta$. Then, using (1.6) and the given assumptions we have

$$\begin{aligned}m(f)(\Delta'')&=(mG)(m(h_1)(\Delta''),M(h_1)(\Delta''),...,m(h_k)(\Delta''),M(h_k)(\Delta''))\\ &\ge(mG)(m(h_1)(\Delta'),M(h_1)(\Delta'),...,m(h_k)(\Delta'),M(h_k)(\Delta'))\\ &=m(f)(\Delta')\end{aligned}$$

In a similar way we can prove that: $M(f)(\Delta'')\le M(f)(\Delta')$. Consequently, the pair $\langle m(f),M(f)\rangle$ is monotone.

Let now, $f,g:D\to\mathbb{R}$ be any two m-M functions. Using Table 1.1 and Lemma 1.3 it is easy to conclude that for their sum $h:D\to\mathbb{R}$

$$h(X):=f(X)+g(X)\qquad (X\in D)$$

one m-M pair can be determined by the following equalities

$$m(h)(\Delta) = m(f)(\Delta) + m(g)(\Delta), \quad M(h)(\Delta) = M(f)(\Delta) + M(g)(\Delta)$$

According to this in the m-M algebra one recursive definition reads

$$(*) \quad m(f+g)(\Delta) = m(f)(\Delta) + m(g)(\Delta), \quad M(f+g)(\Delta) = M(f)(\Delta) + M(g)(\Delta)$$

by which[1] an m-M pair of $f+g$ is defined by the m-M pairs $\langle m(f)(\Delta), M(f)(\Delta)\rangle$ $\langle m(g)(\Delta), M(g)(\Delta)\rangle$ of f and g. Next, denote by

$$(1.7) \qquad Term(\mathbb{R}, x_1, ..., x_n, +, \cdot, -, \sqrt[2k+1]{\ }, \exp, \sin, \cos, \min, \max)$$

the set of all terms built up from[2]

> the variables x_1, ..., x_n, symbols of some real numbers and functional symbols $+, \cdot, -, \sqrt[2k+1]{\ }, \exp, \sin, \cos, \min, \max$ where $k > 0$ may be any natural number

Assuming that f, g can be any such terms and that $\Delta = [\alpha_1, \beta_1] \times ... \times [\alpha_n, \beta_n]$ can be any n-dimensional segment[3] we have the following recursive definition of the so-called <u>*general m − Mpair*</u> (of any term f belonging to set (1.7).

Definition 1.1.

(i) $m(C)(\Delta) = C$, $M(C)(\Delta) = C$, *(C is a constant)*
$m(x_i)(\Delta) = \alpha_i$, $M(x_i)(\Delta) = \beta_i$, $(i = 1, ..., n)$

(ii) $m(f+g)(\Delta) = m(f)(\Delta) + m(g)(\Delta)$, $M(f+g)(\Delta) = M(f)(\Delta) + M(g)(\Delta)$,

(iii) $m(-f)(\Delta) = -M(f)(\Delta)$, $M(-f)(\Delta) = -m(f)(\Delta)$,

(iv) $m(f \cdot g)(\Delta) = \min(m(f)(\Delta)m(g)(\Delta), m(f)(\Delta)M(g)(\Delta), M(f)(\Delta)m(g)(\Delta), M(f)(\Delta)M(g)(\Delta))$
$M(f \cdot g)(\Delta) = \max(m(f)(\Delta)m(g)(\Delta), m(f)(\Delta)M(g)(\Delta), M(f)(\Delta)m(g)(\Delta), M(f)(\Delta)M(g)(\Delta))$

(v) $m(\min(f,g))(\Delta) = \min(m(f)(\Delta), m(g)(\Delta))$
$M(\min(f,g))(\Delta) = \min(M(f)(\Delta), M(g)(\Delta))$

(vi) $m(\max(f,g))(\Delta) = \max(m(f)(\Delta), m(g)(\Delta))$
$M(\max(f,g))(\Delta) = \max(M(f)(\Delta), M(g)(\Delta))$

(vii) $m(\sqrt[2k+1]{f})(\Delta) = \sqrt[2k+1]{m(f)(\Delta)}$, $M(\sqrt[2k+1]{f})(\Delta) = \sqrt[2k+1]{M(f)(\Delta)}$

(viii) $m(\exp f)(\Delta) = \exp m(f)(\Delta)$, $M(\exp f)(\Delta) = \exp M(f)(\Delta)$

(ix) $m(\sin f)(\Delta) = m(f)(\Delta) - M(f)(\Delta) + \sin m(f)(\Delta)$
$M(\sin f)(\Delta) = M(f)(\Delta) - m(f)(\Delta) + \sin M(f)(\Delta)$

(x) $m(\cos f)(\Delta) = m(f)(\Delta) - M(f)(\Delta) + \cos m(f)(\Delta)$

[1]But, it is not true that generally for each m-M pair of the sum $f+g$ exist some m-M pairs for f, g such that equalities (*) hold

[2]The symbol − is taken as an unary functional symbol. According, the difference $x-y$ is introduced as $x + (-y)$.

[3]Thus it is not supposed that $\Delta \subseteq D$.

$$M(\cos f)(\Delta) = M(f)(\Delta) - m(f)(\Delta) + \cos M(f)(\Delta)$$

As it is well-known to each term f belogning to the set (1.7) one can correspond the unique function[4] $f : \mathbb{R}^n \to \mathbb{R}$ in the standard way.

According to the Table 1.1 and Lemma 1.3 the left-hand sides of each of equalities *(ii)-(viii)* determines one m-M pair of the functions $f+g, -f, f \cdot f, \min(f,g)$, $\max(f,g), \exp f$ respectively by means of the m-M pairs of the functions f, g. A similar fact holds for equalities *(ix)*, *(x)* due to the following identities

$$(1.8) \qquad \sin x = (x + \sin x) - x, \qquad \cos x = (x + \cos x) - x$$

by which the functions <u>sine</u> and <u>cosine</u> are represented as differences of two monotone functions.

Note also that repeated application of this definition always produces monotone pairs. This follows from the fact that by part *(i)* of Definition 1.1 one monotone m-M pair is introduced and that other equalities in Definition 1.1 are compatible with monotony property (see Remark 1.1). In such a way we have proved the following lemma.

Lemma 1.4 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function defined by a term $f(x_1, ..., x_n)$ belonging to set (1.7). This function is an m-M function. Employing Definition 1.1 one of its m-M pairs can be effectively found in a finite number*[5] *of steps. The obtained m-M pair is monotone.*

For illustration we give the following examples

Example 1.1. *Let f be a polynomial function defined by the equality of the form*

$$f(x_1, ..., x_n) = g(x_1, ..., x_n) - h(x_1, ..., x_n)$$

where $g(x_1, ..., x_n)$, $h(x_1, ..., x_n)$ are polynomials having positive coefficients only. Then one m-M pair of f is determined by the following equalities

$$m(f)(\Delta) = g(\alpha_1, ..., \alpha_n) - h(\beta_1, ..., \beta_n), \quad M(f)(\Delta) = g(\beta_1, ..., \beta_n) - h(\alpha_1, ..., \alpha_n)$$
$$(\Delta := [\alpha_1, \beta_1] \times ... \times [\alpha_n, \beta_n])$$

providing that $\alpha_1, ..., \alpha_n \geq 0$.

Example 1.2. *According to the identity* $|x| = \max(x, -x)$ *from Definition 1.1 one can easily deduce the following equalities*

$$m(|f|)(\Delta) = \max(m(f)(\Delta), -M(f)(\Delta)), M(|f|)(\Delta) = \max(M(f)(\Delta), -m(f)(\Delta))$$

2. In m-M calculus we shall frequently be concerned with dividing some given segments of reals into certain smaller "pieces". In connection with it we introduce

[4]To be more precise we can denote this function by some new symbol, for instance $\bar{f}$. Then its definition would read

$\bar{f}(x_1, ..., v_n) =$ Value of term $f(x_1, ..., x_n)$ (for any $x_1, ..., x_n \in \mathbb{R}$)

[5]As a matter of fact this number is equal to the number of all functional symbols occuring in the term $f(x_1, ...x_n)$.

the so-called cell-decomposition of a given segment $[a, b] \subset \mathbb{R}$. Any such decomposition $\mathcal{D}$ is an infinite set of certain segments $[a', b'] \subseteq [a, b]$, the so-called cells of the decomposition, where to each cell one of the numbers 0, 1, 2, ..., the so-called order of the decomposition, is ascribed. In addition the following conditions are supposed

(1.9) (i) $[a, b] \in \mathcal{D}$

(ii) For each[6] $r \in \mathbb{N}$ there exists a finite number of cells in $\mathcal{D}$ having the order r. The segment $[a, b]$ is the unique cell of order 0.

(iii) The union of all cell of order r is $[a, b]$.

(iv) The interiors of two different cells of the same order r are disjoint.

(v) If $d(r)$ denotes the maximum of length of all cells of order r the equality

$$\lim_{r \to \infty} d(r) = 0$$

holds[7].

A cell-decomposition $\mathcal{D}$ is called a cell-tree if the following condition is fullfiled

(vi) For each cell $C_r \in \mathcal{D}$ of order $r(> 0)$ there exists a unique cell $C_{r-1} \in \mathcal{D}$ of order $r - 1$ such that $C_r \subseteq C_{r-1}$.

One example of cell-tree is the so-called diadic tree. Its cells of order r are segments $[\alpha, \beta] \subseteq [a, b]$ defined by the equalities of the form

$$\alpha = a + k(b - a) \cdot 2^{-r}, \quad \beta = \alpha + (b - a) \cdot 2^{-r}$$

where k can be any element of the set $\{0, 1, .., 2^r - 1\}$. Notice that by the definition of cell-decomposition for each decomposition $\mathcal{D}$ of the segment $[a, b]$ the following fact holds

(1.10) To each point $x \in [a, b]$ at least one sequence $\langle C_r(x) \rangle$ of r-cells[8] is related such that the following conditions

$$(\forall r \in \mathbb{N})\, x \in C_r(x)$$

is satisfied

Any such sequence is called a cell sequence of x. In the sequel we shall frequently use the following definition

Definition 1.2.

1^0 *Let $\mathcal{D}[a, b]$ be a cell-decomposition of the segment $[a, b] \subseteq \mathbb{R}$. Then the set of all r-cells of the decomposition is denoted by $\mathcal{D}_r[a, b]$.*

2^0 *Let $D = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$ be an n-dimensional segment and let $\mathcal{D}[a_i, b_i]$ be some cell-decompositions of the segments $[a_i, b_i]$ $(i = 1, ..., n)$.*

[6] $\mathbb{N}$ is the set of all nonnegative integers 0, 1, 2, . . .

[7] Obviously the notion of decomposition is that appearing in the ordinary definition of Reimann integral.

[8] Instead of "cell of order r" we say briefly "r-cell".

Then the sets $\mathcal{D}_r(D)$, $\mathcal{D}(D)$ (r is a fixed element of $\mathbb{N}$) are introduced by the following equalities

$$\mathcal{D}_r(D) = \{P_1 \times ... \times P_n | P_1 \in \mathcal{D}_r[a_1, b_1], ..., P_n \in \mathcal{D}_r[a_n, b_n]\}$$
$$\mathcal{D}(D) = \cup_{r \in \mathbb{N}} \mathcal{D}_r(D)$$

respectively.

3. Let us now extend the set (1.7) by allowing the set of functional symbols to contain also the following new symbols

$$1/, \ \text{arcsin}, \ \ln, \ \sqrt[k]{\ }, \quad (k > 1, k \in \mathbb{N})$$

This new term set will be denoted by

(1.11) $\quad Term(\mathbb{R}, x_1, ..., x_n, +, \cdot, -, \exp, \sin, \cos, \min, \max, 1/, \text{arcsin}, \ln, \sqrt[k]{\ })$

If $t(x_1, .., x_n)$ is any element of this set then in the standard way one can correspond to it one function. But, in general such a function is defined only for those values $(x_1, ..., x_n) \in \mathbb{R}^n$ which satisfy the corresponding definition-condition *Cond(t)*, to be defined bellow.

Denote by $P = \{p_1, ..., p_a\}$ the set of all terms such that $1/p_1$, ..., $1/p_a$ are all subterms of the term t having the form $1/p$, where p is a term. Similarly let

$$Q = \{q_1, ..., q_b\}, \quad R = \{r_1, ..., r_c\}, \quad S = \{s_1, ..., s_d\}$$

be the sets of all terms such that

$$\text{arcsin} q_1, ..., \text{arcsin} q_b, \ \ln r_1, ..., ln r_c, \ \sqrt[2k]{s_1}, ..., \sqrt[2k]{s_d} \quad (k > 0, k \in \mathbb{N})$$

are all subterms of the term t having the form arcsin$q$, $\ln r$, $\sqrt[2k]{s}$ respectively. Some of these sets P, Q, R, S may be empty. Then we have the following definition

(1.12) *Cond(t) is the conjuction of the following conditions*

$$p_1 \neq 0, ..., p_q \neq 0, |q_1| \leq 1, ..., |q_b| \leq 1, r_1 > 0, ..., r_c > 0, s_1 \geq 0, ..., s_d \geq 0$$

Denote by *Dom(t)* the set of all $X \in \mathbb{R}^n$ satisfying the condition *Cond(t)*.

So, the following problem:

(1.13) $\quad$ Determine the set *Dom(t)*

appears.

A bit later we shall consider that problem under the following additional condition

$$(x_1, ..., x_n) \in D$$

where, as before $D = [a_1, b_1] \times ... \times [a_n, b_n] \subseteq \mathbb{R}^n$ is n-dimensional segment.

Now bearing in mind the corresponding definition-condition about the new functional symbols we extend Definition 1.1 by this addendum

Definition 1.3.

(*xi*) $m(1/f)(\Delta) = 1/M(f)(\Delta),\ M(1/f)(\Delta) = 1/m(f)(\Delta),$
$$\text{if } 0 \notin [m(f)(\Delta), M(f)(\Delta)]$$

(*xii*) $m(arcsinf)(\Delta) = arcsinm(f)(\Delta),\ M(arcinf)(\Delta) = arcsinM(f)(\Delta),$
$$\text{if } -1 \le m(f)(\Delta) \text{ and } M(f)(\Delta) \le 1$$

(*xiii*) $m(\ln f)(\Delta) = \ln m(f)(\Delta),\ M(\ln f)(\Delta) = \ln M(f)(\Delta),$
$$\text{if } m(f)(\Delta) > 0$$

(*xiv*) $m(\sqrt[2k]{f})(\Delta) = \sqrt[2k]{m(f)(\Delta)},\ M(\sqrt[2k]{f})(\Delta) = \sqrt[2k]{M(f)(\Delta)},\quad (k > 0, k \in \mathbb{N})$
$$\text{if } m(f) \ge 0$$

Bearing in mind the conditional character of Definition 1.3 one must be careful in using it.

Suppose now that $\Delta = [\alpha_1, \beta_1] \times \dots \times [\alpha_n, \beta_n] \subset \mathbb{R}^n$ is any n-dimensional segment. not necessary a subset of D, and that f is a function defined by certain term belonging to set (1.11). By means of Definition 1.1 and 1.3 we can recursively, step-by-step, try to determine a pair of numbers $m(f)(\Delta)$, $M(f)(\Delta)$. With more details this procedure can be described as follows.

(1.14) We start by using equalities (i) and after that, if it is needed, according to the structure of the term of the function f we step-by-step use one of the equalities[9] (ii), (iii), ..., (xiv). Then, whenever we meet some subterm of one of the form
$$1/t, \text{arcsin}t, \ln t, \sqrt[2k]{t}, \quad (k > 0, k \in \mathbb{N})$$
we first check whether the corresponding condition from Definition 1.3, i.e. one of these

(c_1) $0 \notin [m(t)(\Delta), M(t)(\Delta)]$, (c_2) $-1 \le m(t)(\Delta)$ and $M(t) \le 1$,
(c_3) $m(t)(\Delta) > 0$, (c_4) $m(t)(\Delta) \ge 0$

respectively is satisfied.

The procedure halts if that condition, let us say (c_i), is not satisfied. In such a case we shall say that the procedure has stoped *prematurely*. At this halting step we also check whether the following additional condition (c_i'):

(c_1') $m(t)(\Delta) = M(t)(\Delta) = 0$
(c_2') $m(t)(\Delta) > 1$ or $M(t)(\Delta) < -1$
(c_3') $M(t)(\Delta) \le 0$
(c_4') $M(t)(\Delta) < 0$

respectively is satisfied.

In the connection with this procedure we introduce the following definitions[10].

[9] As a matter of fact this procedure is quite similar to that of computing the value of some given algebric expression in which eventually some operators are only partialy defined.

[10] We use the terms **f-solutional**, **f-indetermined**, **f-feasible** because of the similarity with the notions **solutional**, **indetermined**, **feasible** (see Definitions 2.1, 2.2)

Definition 1.4. *An n-dimensional segment Δ is*

1^0 *f-solutional if the procedure (1.14) does not halt prematurely*

2^0 *f-indermined if the procedure (1.14) halts prematurely but then the corresponding condition (c_i') is not satisfied.*

Definition 1.5. *An n-dimensional segment Δ is f-feasible if it is f-solutional or f-indetermined.*

Now obviously we have the following:

Lemma 1.5. *Let Δ be any n-dimensional. Then:*

(*i*) *If Δ is an f-solutional then the function f is defined for all $X \in \Delta$.*
(*ii*) *If Δ is not f-feasible then for all $X \in \Delta$ the function f is not defined.*

The proof follows immediately from the meaning of conditions (c_i), (c_i') $(i = 1, 2, 3, 4)$.

Lemma 1.6. *Let Δ and Δ', with $\Delta' \subseteq \Delta$, be any n-dimensional segments. Then if Δ is f-solutional Δ' is f-solutional too.*

Proof. Denote by $o(f)$ the number of all functional symbols occuring in the term f. To prove the lemma we shall prove by induction on $o(f)$ the following implication

(*1) *If Δ is a f-solutional and $\Delta' \subseteq \Delta$ then Δ' is f-solutional too, and additionally the inequalities*
$$m(f)(\Delta) \le m(f)(\Delta'), \quad M(f)(\Delta') \le M(f)(\Delta)$$
are satisfied.

If $o(f) = 0$ then the form f is one of the terms $x_1, \dots, x_n, C$, where C is a constant and assertion (*1) is trivially true.

If $o(f) > 0$ then the term f can have one of the following forms

1^0 $u + v$, 2^0 $u \cdot v$, 3^0 $-u$, 4^0 $\sqrt[2k+1]{u}$, 5^0 $\exp u$, 6^0 $\sin u$, 7^0 $\cos u$
8^0 $\min(u, v)$, 9^0 $\max(u, v)$, 10^0 $1/u$, 11^0 $\text{arcsin}u$, 12^0 $\ln u$, 13^0 $\sqrt[2k]{u}$

where u, v are some terms and $k > 0$, $k \in \mathbb{N}$.

Let f have the form $u + v$. Then using the induction hypothesis and Remark 1.1 it follows that (*1) holds. Similarly one can treat cases 2^0-9^0.

Let f have the form $1/u$. Then by the induction hypothesis we have the following conditions

(*2)
$$m(u)(\Delta) \le m(u)(\Delta'), \quad M(u)(\Delta') \le M(u)(\Delta)$$

By assumption we also have the condition
$$o \notin [m(u)(\Delta), M(u)(\Delta)]$$
i.e. the following condition

(*3)
$$m(u)(\Delta) > 0 \lor M(u)(\Delta) < 0$$

From (*2) and (*3) it follows that

(*4) $$m(u)(\Delta') > 0 \vee M(u)(\Delta') < 0$$

which completes proof in case 10^0. In a similar way the remaining cases can be proved (see also Remark 1.2 below).

A straigtforward corrolary of Lemma 1.6 reads

(1.15) *Let Δ be an f-solutional n-interval. Then using procedure (1.14) one can for each $\Delta' \subseteq \Delta$ determine the m-M pair $\langle m(f)(\Delta'), M(f)(\Delta')\rangle$. Moreover, this m-M pair is monotone.*

Note that this assertion is a generalization of Lemma 1.4.

Remark 1.2. *Denote condition (*3) by $\varphi(\Delta)$. Then the implication (*3) $\Rightarrow$ (*4) can be rewritten as follows*

$$\varphi(\Delta) \Rightarrow \varphi(\Delta') \quad (\textit{If } \Delta \subseteq \Delta' \textit{ and } (*2))$$

In other words the condition $\varphi(\Delta)$ is **compatible** *with the monotony property. The similar fact holds for conditions (c_2), (c_3), (c_4).*

Let now t be a term belonging to set (1.11) and let $D \subset \mathbb{R}^n$ be a given n-dimensional segment. We are going to solve problem (1.13) under the assumption $(x_1, ..., x_n) \in D$. The set $D \cap Dom(t)$ will be denoted by $Dom_D(t)$.

We start by choosing some decomposition $\mathcal{D}(D)$. (see Definition 1.2). Accordingly let, for fixed r, $F_r(t)$ denote the union of all t-feasible products $P_r \in \mathcal{D}_r(D)$. Next, by $Sing_D(t)$ denote the set of all values $(x_1, ..., x_n) \in D$ at which, if t has one of the form

$$1/u(x_1, ..., x_n), \quad \ln u(x_1, ..., x_n)$$

the condition of the form

$$u(x_1, ..., x_n) = 0$$

is satisfied. Then we have the following basic result.

Theorem 1.1. *If t is any term from set (1.11) then the following equality*

$$\left(\bigcap_{r \in \mathbb{R}} F_r(t)\right) \backslash Sing_D(t) = Dom_D(t)$$

holds.

Proof. Let first $X = (x_1, ..., x_n) \in Dom_D(t)$. This point does not belong to $Sing_D(t)$. Next, denote by P_r the direct product $C_r(x_1) \times ... \times C_r(x_n)$ where $\langle C_r(x_i)\rangle$ is any cell-sequence of the number x_i $(i = 1, 2, ..., n)$. Suppose that for some $r \in \mathbb{N}$ the product P_r is not feasible. Then, by Lemma 1.5, (ii) the function f, corresponding to the term t, is not defined for all elements of P_r, which contradicts $X \in Dom_D(t)$. Thus, for every $r \in \mathbb{N}$ the products P_r must be f-feasible. Consequently: $X \in \left(\bigcap_{r\in\mathbb{N}} F_r\right) \backslash Sing_D(t)$.

Let now $X \in \bigcap_{r\in\mathbb{N}} F_r$ and $X \notin Sing_D(t)$. To prove $X \in Dom_D(t)$ suppose the contrary. Then the term t must have some subterm of one of the forms

$$\arcsin u, \quad \sqrt[2k]{u}, \quad \ln u \qquad (k > 0, k \in \mathbb{N})$$

and additionally the following condition

$$|u(X)| > 1, \quad u(X) < 0, \quad u(X) < 0$$

respectively is fulfilled. Consequently for r great enough any product $P_r \ni X$ will satisfy the condition

$$m(u)(P_r) > 1 \text{ or } M(u)(P_r) < -1, \quad M(u)(P_r) < 0, \quad M(u)(P_r) < 0$$

respectively. Thus P_r is not feasible, which contradicts $X \in \bigcap F_r$. The proof is completed.

Remark 1.3. *The set $Dom_D(t)$ is the set of all solutions of the condition $Cond(T)$ (see 1.12). Accordingly, applying the procedure (2.6) from section 2 one can aproximately determine this set as follows. Let $\varepsilon > 0$ be a real number chosen in advance and let $Cond_\varepsilon(t)$ be the conjuction of the following conditions*

$$|p_1| \geq \varepsilon, ..., |p_a| \geq \varepsilon, \quad |q_1| \leq 1, ..., |q_b| \leq 1, \quad |r_1| \geq \varepsilon, ..., |r_c| \geq \varepsilon$$

$$s_1 \geq 0, ..., s_d \geq 0, (x_1, ..., x_n) \in D$$

$Cond(t)$ is a system of inequalities to which Theorem 2.1 can be applied. Consequently the set $Dom_D(t)$ can be approximately determined in that way.

In connection with the mentioned procedure (2.6) here we add the following remark.

Remark 1.4. *If in the system (2.1) any function f_i is defined by some term from the set (1.11) then, briefly said, in Definition 2.1 the feasibilty criterion should include the following part: Δ is f_i-solutional. Also, those Δ's who are f_i-indetermined should be included to U_r, but as a separate part. The reason is: such Δ's have to be treated in the next step of the procedure.*

4. Suppose now that f is a function defined by a term $f(x_1, ..., x_n)$ belonging to set (1.11) and in some points S the value of that term becomes indetermined (sych as $0/0$, ∞/∞, etc.). Usually, $f(S)$ is defined as the corresponding limit (if it exists). The function $f : D \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} \frac{(x_1^3)}{(x_1^2 + x_2^2)} & \text{if } (x_1, x_2) \neq (0,0) \\ 0 & \text{if } (x_1, x_2) = (0,0) \end{cases}$$

$$S = (0,0), \quad D \subseteq \mathbb{R}^2, \quad D \ni S$$

is an example of such a function. In the case of such functions in order to determine one m-M pair $\langle m(f)(\Delta), M(f)(\Delta)\rangle$, where $\Delta \ni S$ is a domain small enough, one can apply Lemma 1.7 stated below.

Lemma 1.7. *Let $f : D \to \mathbb{R}$ be a given function and $S \in D$. Suppose that there exist a number ε, with*[11] $0 < \varepsilon < \min_{X\in\partial D} \| X - S \|$, *and a function $\lambda : [0,\varepsilon] \to \mathbb{R}$ having the following properties*

(j) λ *is a non-negative function*

(jj) $\lim_{t\to+0} \lambda(t) = 0$

(jjj) $\| X - S \| \le \varepsilon \Rightarrow |f(X) - f(S)| \le \lambda(\| X - S \|)$

Then for all segments $\Delta \subseteq D$ with the properties $S \in \Delta$, $diam\Delta \le \varepsilon$ one m-M pair of the function f is determined by the equalities

(1.16) $\quad m(f)(\Delta) = f(S) - \lambda(diam\Delta),\ M(f)(\Delta) = f(S) + \lambda(diam\Delta)$

Proof is straightforward.

Notice that the condition (jjj) is weaker then the Lipschitz condition which involves a function λ of the form $\lambda(t) = Kt^\alpha$ where $K, \alpha > 0$ are constants. The condition (jjj) is very natural since for each continuous function $f : D \to \mathbb{R}$, $S \in D$ one can easily prove the existence of $\varepsilon > 0$ and a function λ with the properties (j), (jj), (jjj).

In connection with the function f defined above we mention that according to the inequality

$$|x_1^3 + x_2^3| \le 2r^3 \text{ where } r^2 = x_1^2 + x_2^3$$

one can define a function λ by the following equality $\quad \lambda(t) = 2|t|$.

5. Now we are going to give some formulas for m-M pairs in case of differentiable functions (and complex regular functions).

Theorem 1.2. *Let $f : [a_1, b_1] \to \mathbb{R}$ be a given function belonging to the class $C^{k+1}[a_1, b_1]$ where k is some natural number. Suppose also that for any segment $\Delta = [\alpha_1, \beta_1]$ (with $\Delta \subseteq [a_1, b_1]$)*

(1.17) $$B(|f^{(k+1)}|)(\Delta)$$

denotes an upper bound of the modules of the $(k+1)$-derivative of[12] *f when $x \in \Delta$. Additionally suppose that the following condition is satisfied*

(1.18) $$(\forall\varepsilon > 0)\,(\exists K \in \mathbb{R})\left(diam\Delta < \varepsilon \Rightarrow B(|f^{(k+1)}|)(\Delta) < K\right)$$

that is (1.17) is bounded if $diam\Delta \to 0$. Then, using notations

$$\gamma = \frac{\alpha_1 + \beta_1}{2}, \quad \rho = \frac{\beta_1 - \alpha_1}{2}$$

one m-M pair of the function f is determined by the following equalities

(1.19)
$$m(f)(\Delta) = f(\gamma) - \sum_{i=1}^{k} \frac{|f^{(i)}(\gamma)|}{i!}\rho^i - \frac{\rho^{k+1}}{(k+1)!}B(|f^{(k+1)}|)(\Delta)$$
$$M(f)(\Delta) = f(\gamma) + \sum_{i=1}^{k} \frac{|f^{(i)}(\gamma)|}{i!}\rho^i + \frac{\rho^{k+1}}{(k+1)!}B(|f^{(k+1)}|)(\Delta)$$

[11] ∂D denotes boundary of D

[12] It suffices that x belongs to the interior of Δ only.

Proof. Let us start with the following well-known identity

(*1) $$f(x) = f(\gamma) + \sum_{i=1}^{k} \frac{(x-\gamma)^i}{i!} f^{(i)}(\gamma) + \frac{1}{k!}\int_\gamma^x f^{(k+1)}(t)(x-t)^k dt$$

further for the above integral we have

(*2)
$$\left|\int_\gamma^x f^{(k+1)}(t)(x-t)^k dt\right| = \left|\int_1^0 f^{(k+1)}(x + t(\gamma - x))(-t(\gamma - x))^k(\gamma - x)dt\right|$$
$$\le \rho^{k+1} B\left(\left|f^{(k+1)}\right|\right)(\Delta)\int_0^1 t^k dt$$
$$= \frac{\rho^{k+1}}{k+1} B\left(\left|f^{(k+1)}\right|\right)(\Delta)$$

Using (*1) and (*2) it is easy to prove the formulas (1.19), i.e. to complete the proof.

Definition 1.6. *The m-M pair defined by (1.19) is called the k-Taylor m-M pair of the function f.*

Remark 1.5. *In formula (1.19) one may take for γ any element of Δ, but then ρ should be $diam\Delta$. Similar facts hold for formulas (1,25) and (1,26) below.*

It is interesting that the formulas (1.19) can be generalize to the case of real functions in several variables and also to the case of complex regular functions.

For instance, if $f : D \to R$, where $D = [a_1, b_1] \times [a_2, b_2]$, is a real function belonging to the class $C^2(D)$ then similarity to (1.19) one pair of f is determined by the following equalities

(1.20)
$$m(f)(\Delta) = f(\gamma_1, \gamma_2) - \left|\frac{\partial f}{\partial x_1}(\gamma_1, \gamma_2)\right|\rho_1 - \left|\frac{\partial f}{\partial x_2}(\gamma_1, \gamma_2)\right|\rho_2 - \frac{B}{2}$$
$$M(f)(\Delta) = f(\gamma_1, \gamma_2) + \left|\frac{\partial f}{\partial x_1}(\gamma_1, \gamma_2)\right|\rho_1 + \left|\frac{\partial f}{\partial x_2}(\gamma_1, \gamma_2)\right|\rho_2 + \frac{B}{2}$$

$$\left(\gamma_i = \frac{\alpha_i + \beta_i}{2}, \quad \rho_i = \frac{\beta_i - \alpha_i}{2}, \quad B = \rho_1^2 B_{11} + 2\rho_1\rho_2 B_{12} + \rho_2^2 B_{22}\right)$$

where B_{11}, B_{12}, B_{22} are respectively

$$B\left(\left|\frac{\partial^2 f}{\partial x_1^2}\right|\right)(\Delta), \quad B\left(\left|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right|\right)(\Delta), \quad B\left(\left|\frac{\partial^2 f}{\partial x_2^2}\right|\right)(\Delta),$$

providing that the conditions of the type (1.18) are satisfied.

Notice that the formulas (1.20) may be easily proved by using identity[13] (*1) in the case of the following auxiliary function $g : [0, 1] \to R$

$$g(t) = f(\gamma_1 + t(x_1 - \gamma_1), \gamma_2 + t(x_2 - \gamma_2)) \quad (x_1, x_2 \text{ are fixed})$$

[13] with $k = 2$

The formulas (1.20) determine one 1-Taylor pair of a given function. Of course, generally k-Taylor pair, with $k \in \mathbb{N}$, can be defined in quite similar way.

Let now[14] $f : D \to \mathbb{C}$, $D = [a_1, b_1] \times [a_2, b_2]$ be a given complex regular function and k some natural number. Further, let

(*3) $\Delta = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$, $\gamma = \frac{\alpha_1+\alpha_2}{2} + i\frac{\beta_1+\beta_2}{2}$, $2\rho = \left((\beta_1 - \alpha_1)^2 + (\beta_2 - \alpha_2)\right)^{1/2}$

Then similarly to (1.19) one k-Taylor m-M pair of the function $|f(z)|$ can be detrmined by the following equalities

$$(1.21)\qquad \begin{aligned} m(|f|)(\Delta) &= |f(\gamma)| - \sum_{i=1}^{k} \frac{|f^{(i)}(\gamma)|}{i!}\rho^i - \frac{\rho^{k+1}}{(k+1)!}B_{k+1} \\ M(|f|)(\Delta) &= |f(\gamma)| + \sum_{i=1}^{k} \frac{|f^{(i)}(\gamma)|}{i!}\rho^i + \frac{\rho^{k+1}}{(k+1)!}B_{k+1} \end{aligned}$$

where B_{k+1} denotes the upper bound

$$B\left(\left|f^{(n+1)}\right|\right)(\Delta)$$

supposing that the condition of the type (1.18) is satisfied.
Formulas (1.21) can be proved quite similarly as formulas (1.19). Namely, we can again start with the identity of the form (*1) proceed as in the proof of (1.19) but now we must use in the notion of the modulus of complex numbers. In connection with it let us point out that in such a way we first obtain the following formula

$$(1.22)\qquad ||f(z)| - |f(\gamma)|| \le \sum_{i=1}^{k} \frac{|f^{(i)}(\gamma)|}{i!}|z-\gamma|^i + \frac{|z-\gamma|^{k+1}}{(k+1)!}B_{k+1}$$

from which formulas (1.21) follow immediately. Let us write the inequality (1.22) in the following form

$$(1.23)\qquad ||f(z)| - |f(\gamma)|| = o(1) \qquad (\text{if } |z-\gamma| \to 0)$$

emphasizing that the right-hand side of (1.22) has limit 0 when $|z-\gamma|$ tends to 0. It is interesting that such an inequality may be valid for certain complex functions which are not regular. So, for function $f(z) = |z|$ we have inequality

$$(\Delta)\qquad ||z| - |\gamma|| \le |z - \gamma|$$

which directly yields a fact of the form (1.23). According to this, using (1.22) and (Δ) we can find an m-M pair for a complex function f which is built up a certain composition of some regular complex functions and the function $|z|$. To illustrate

[14] Complex numbers are treated as ordered pairs of real numbers.

this we give an example. Let f be the complex function defined by the equality $f(z) = e^z \sin|z|$. Then we have:[15]

$$\begin{aligned} f(z) &= (e^\gamma + (z-\gamma)A)\cdot(\sin|\gamma| + B\cdot(z-\gamma)) \\ &\quad \left(\text{for some } A, B \text{ satisfying the inequalities}^{16)}\ |A| \le e^{\alpha_2}, |B| \le 1\right) \\ &= e^\gamma \sin|\gamma| + (z-\gamma)\left[AB(z-\gamma) + A\sin|\gamma| + Be^\gamma\right] \end{aligned}$$

wherefrom one can easily make an m-M formula for f.

6. Now we turn our attention to Lemma 1.1. Obviously, in practice we have the ideal case if we may defined an m-M pair of $f : D \to \mathbb{R}$ by equalities of this lemma, i.e. by:

(1.24) $m(f)(\Delta) = \min\{f(V_1), ..., f(V_{2^n})\}$, $M(f)(\Delta) = \max\{f(V_1), ..., f(V_{2^n})\}$
($V_1, ..., V_{2^n}$ are all vertices of $\Delta \subseteq D$)

This m-M pair will be called the **ideal** m-M pair. Then the segment $\Delta \subseteq D$ will similarly be called the f-ideal segment. For the ideal m-M pair besides the denotations $m(f)(\Delta)$, $M(f)(\Delta)$ we may use the simbols

$$\min(f)(\Delta), \max(f)(\Delta),$$

respectively.
Thus far we have used formulas (1.24) several times (see Table 1.1). In general the formulas can be used in case of monotone functions (with respect to each of its arguments). For instance, we have such a case if:

(1.25) f is a linear function.

Next, suppose that the function $f(x_1, ..., x_n)$ has the derivatives

$$\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n} \qquad (\text{for all } (x_1, ..., x_n) \in \Delta)$$

and additionally

(1.26)Each $\frac{\partial f}{\partial x_i}$ $(i = 1, 2, .., n)$ has a fixed sign ($\le$ or $\ge$) for all $(x_1, ..., x_n) \in$ Interrior (Δ).

Then Δ, and also any subsegment $\Delta' \subset \Delta$, are obviously f-ideal segments.

7. Now we state some general properties of m-M pairs. Let $f : D \to \mathbb{R}$ be an m-M function and let $T(D)$ be some, by Definition 1.2, chosen cell-decomposition which is a cell-tree. Next, let $\langle m(f)(\Delta), M(f)(\Delta)\rangle$ with $\Delta \subseteq D$ be a given m-M pair of the function f. We introduce the following denotation

(1.27)*For given $r \in \mathbb{N}$ and $P_r \in T_r(D)$ by P_r' is denoted the unique $(r-1)$-product containing P_r as a subset.*

[15] We recall that $z \in D$ and γ is defined by (*3)

[16] For instance, (Δ) implies an equality of the form $|z| = |\gamma| + B\cdot(z-h)$ with some B satisfying the inequality $|B| \le 1$.

Thus, $P'_r \in T_{r-1}(D)$ and $P'_r \supseteq P_R$.

Definition 1.7. *For all elements $P_r \in T(D)$ an m-M pair of the function f, denoted by $m^*(f)(P_r)$, $M^*(f)(P_r)$, is defined recursively by these equalities*

$$m^*(f)(P_0) = m(f)(P_0), \quad M^*(f)(P_0) = M(f)(P_0) \qquad (P_0 \text{ is } D)$$
$$m^*(f)(P_{r+1}) = \max(m(f)(P_r), m^*(f)(P'_r)),$$
$$M^*(f)(P_{r+1}) = \min(M(f)(P_r), M^*(f)(P'_r))$$

In virtue of this definition it follows immediately:

(1.28) *The $m^*(f)(\Delta)$, $M^*(f)(\Delta)$ (with $\Delta \in T(D)$) are monotone (in sense (1.2)).*

We emphasize that this elementary fact can be very suitable used in various applications of m-M calculus (see, for instance, Remark 3.1).

Lemma 1.8. *Let $f : D \to \mathbb{R}$ be an m-M function and let for each n-subsegment $\Delta \subseteq D$ one point X_Δ be choosen in advance. Then the following relations are true*

(*) $m(f)(\Delta) = f(X_\Delta) + o(1), \quad M(f)(\Delta) = f(X_\Delta) + o(1), \quad$ (*if* $diam\Delta \to 0$)

Proof follows at once from axiom (0.2), i.e. from the fact that

(**) $$M(f) - m(f)(\Delta) = o(1) \quad (diam\Delta \to 0)$$

and axiom (0.1), i.e. the inequalities

$$|m(f)(\Delta) - f(X_\Delta)|, |M(f)(\Delta) - f(X_\Delta)| \leq |M(f)(\Delta) - m(f)(\Delta)|$$

The following assertion is an immediate consequence of Lemma 1.8:

(1.29) *If $f : D \to \mathbb{R}$ is an m-M function then for each m-M pair of f there exist positive numbers ε, K such that $|m(f)(\Delta)| \leq K$, $|M(f)(\Delta)| \leq K$ whenever $\Delta \subseteq D$ and $diam\Delta \leq \varepsilon$.*

Definition 1.8. *Let $\langle m(f), M(f)\rangle$ be an m-M pair of some m-M function $f : D \to \mathbb{R}$. This pair is the so-called Lipschitz's m-M pair if there exist positive numbers ε, K, σ such that the inequality*

$$|M(f)(\Delta) - m(f)(\Delta)| \leq K(diam\Delta)^\sigma \quad (\Delta \subseteq D)$$

is fulfilled whenever $diam\Delta \leq \varepsilon$. More precisely said such an m-M pair is a σ-Lipschitz's m-M pair.

Note that in general we may impose on σ the restriction $\sigma \leq 1$. Namely, in case $\sigma > 1$ the function f must be a constant function. Next, according to Definition 1.6 any Taylor's m-M pair is 1-Lipschitz's m-M pair.

Remark 1.6. *In case of σ-Lipschitz's m-M pairs the relations (*) from Lemma 1.8 may be replaced by these*

$$m(f)(\Delta) \geq f(X_\Delta) - K(diam\Delta)^\sigma, \quad M(f)(\Delta) \leq f(X_\Delta) + K(diam\Delta)^\sigma$$

In the following theorem, supposing that a function f is defined by a term $f(x_1, ..., x_n)$ from set (1.11), one uses a notion of **the strictly f-solutional segment**:

(1.30) *An n-dimensional segment Δ is strictly f-solutional if the following condition is fulfilled:*

whenever the term $f(x_1, ..., x_n)$ has some subterm one of the forms

(i) $1/t$; *(ii)* $\arcsin t$, *(iii)* $\ln t$; *(iv)* $\sqrt[2k]{t}$ $(k > 0, k \in \mathbb{N})$

then the corresponding condition

(i') $0 \notin [m(t)(\Delta), M(t)(\Delta)]$ *(ii')* $-1 < m(t)(\Delta), M(t)(\Delta) < 1$

(iii') $m(t)(\Delta) > 0$, *(iv')* $m(t)(\Delta) > 0$

is satisfied respectively.

Theorem 1.3. *Let $f : D \to \mathbb{R}$ be a function defined by a term $f(x_1, ..., x_n)$ belonging to set (1.11) and let D be an f-solutional segment. Then:*

1^0 *The function f is continuous for every $X \in D$.*

2^0 *Using procedure (1.14) one can for every n-dimensional subsegment $\Delta \subseteq D$ determine the numbers $m(f)(\Delta)$, $M(f)(\Delta)$. The obtained m-M pair is monotone. It is also 1-Lipschitz's m-M pair provided that D is a strictly[17] f solutional segment.*

3^0 *Functions (mf), (Mf) defined by equalities of the form (1.5) are continuous.*

Proof is by induction on on $o(f)$ - the number of all operation symbols

$$+, \cdot, -, \exp, \sin, \cos, \min, \max, 1/, \arcsin, \ln, \sqrt[k]{}$$

occuring in the term $f(x_1, ..., x_n)$.

1^0. In virtue of Lemma 1.5 (i) the function f is defined for all $X \in D$. By induction on $o(f)$ it can be easily proved that the function f is continuous. For instance, during such a proof one case to be considered is that in which the term $f(x_1, ..., x_n)$ has the form $1/t(x_1, ..., x_n)$. According to condition (c_1) in procedure (1.14) one of the inequalities

$$(i)\ m(t)(D) > 0, \quad (ii)\ M(t)(D) < 0$$

holds. If (i) or (ii) is true then for all $X \in D$ the inequality $t(X) \neq 0$ is satisfied, which together with the induction hypothesis yields that the function f is continuous.

2^0 Bearing in mind Lemma 1.6 and (1.15) the only thing to be proved is that the obtained m-M pair $\langle m(f)(\Delta), M(f)(\Delta)\rangle$ is Lipschitz's under condition that the segment D is strictly f-solutional. Again one can employ the induction on $o(f)$. Since the function $|f|$ is continuous, it must be bounded. Using this fact and (1.29) (applied on D) one can in a standard way complete the induction proof. For instance, if during the proof we have the case when the term $f(x_1, ..., x_n)$ has the form

(*) $$\sqrt[2k]{t} \qquad \text{(where } k > 0, k \in \mathbb{N})$$

[17] see (1.30)

Then for any $\Delta \subseteq D$ we have

$$|M(f)(\Delta) - m(f)(\Delta)| = \left| \sqrt[2k]{M(t)(\Delta)} - \sqrt[2k]{m(t)(\Delta)} \right|$$
$$= \frac{1}{2k}\xi^{-1+\frac{1}{2k}} \cdot |M(t)(\Delta) - m(t)(\Delta)|$$
for some $\xi \in ((m(t)(\Delta), M(t)(\Delta))$
$$\leq \frac{1}{2k}c^{-1+\frac{1}{2k}} \cdot |M(t)(\Delta) - m(t)(\Delta)|$$
where $c = m(f)(D) > 0$

(In virtue of (1.15) we have $\xi \geq c$)

wherefrom it is easy to complete the proof in the case when the term $f(x_1, ..., x_n)$ has the form (*).

3^0 Let $\Delta = [\alpha_1, \beta_1] \times ... \times [\alpha_n, \beta_n] \subseteq D$ be any subsegment of D. The (mf), (Mf) are functions on $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n$ with $\Delta \subseteq D$. Again we use induction $o(f)$.

If $o(f) = 0$ then f is x_i, for some i $(i \leq i \leq n)$ and (mf), (Mf) are reduced to α_i, β_i respectively. Obviously (mf), (Mf) are continuous functions at any point $(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n)$ where $\Delta \subseteq D$.

Let $o(f) > 0$. There are 13 subcases related to points (ii), (iii), ..., (xiv) of Definition 1.1 and Definition 1.3.

Case (ii), i.e. f has the form $g + h$. Then using (ii) in Definition 1.1 and the induction hypothesis for g and h we conclued that (mf), (Mf) are continuous functions. As a matter of fact, we used the fact that the sum of two continuous functions is also a continuous function. The proof is similar in cases related to parts (iii), (iv), ..., (x) of Definition 1.1.

Case (xi) i.e. f has the form $1/g$. Then $(mf)(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n)$ is equal to $1/M(g)(\Delta)$, but under condition $M(g)(\Delta) \neq 0$. Consequently using induction hypothesis we conclude that (mf) is a continuous function (at any point $(\alpha_1, ..., \alpha_n,$ $\beta_1, ..., \beta_n)$ with $\Delta \subseteq D$). In a similar way one concludes that (Mf) is also a continuous function.

The remaining subcases, related to (xii), (xiii), (xiv), can be treated in a similar way, therefore the proof is omitted.

2. SYSTEM OF EQUATIONS, SYSTEM OF INEQUALITIES

Supposing that $f_i : D \to \mathbb{R}$ $(i = 1, 2, ..., k)$ are any m-M functions , where $D = [\alpha_1, \beta_1] \times ... \times [\alpha_n, \beta_n]$, we state a method of solving the system of inequalities

$$f_i(x_1, ..., x_n) \geq 0, \quad (x_1, ..., x_n) \in D \qquad (i = 1, 2, ..., k)$$

which includes the case of system of equations. Several convergence questions are studied too.

1. Let $D = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$ be a given n-dimensional segment and let $f_i : D_i \to \mathbb{R}$ $(i = 1, ..., k)$ be given m-M functions.In connection with them we consider the following system of inequalities

(2.1) $\quad f_1(x_1, ..., x_k) \geq 0, ..., f_k(x_1, ..., x_n) \geq 0, \quad$ (assuming $(x_1, ..., x_n) \in D$)

Denote by S the set of all its solutions. In order to determine the set S we shall start with some cell-decomposition $\mathcal{D}(D)$ (see Definition 1.2).

Assume for a moment that $\Delta = [\alpha_1, \beta_1] \times ... \times [\alpha_n, \beta_n]$ is any n-dimensional subsegment of D. Generally such a segment can satisfy just one of the following conditions[1)]

$$1^0\ (\forall i)\ M(f_i)(\Delta) \geq 0, \quad 2^0\ (\exists i)\ M(f_i)(\Delta) < 0$$

Obviously a segment Δ satisfying condition 2^0 cannot contain any solution of the system (2.1). Accordingly, we introduce the following definition.

Definition 2.1. *An n-dimensional segment $\Delta \subseteq D$ is* **feasible** *in the sense of system (2.1) if the following condition*

$$(\forall i)\ M(f_i)(\Delta) \geq 0$$

is satisfied.

To this definition we add the following obvious assertion

(2.2) *If an n-dimensional segment $\Delta \subseteq D$ contains a solution of system (2.1) than Δ must be a feasible segments.*

In connection with $\mathcal{D}(D)$ for a fixed $r \in \mathbb{N}$ denote by F_r the union of all feasible segments belonging to $\mathcal{D}_r(D)$. Then our first result about system (2.1) reads:

Theorem 2.1. *The equality*

$$S = \bigcap_{r \in \mathbb{N}} F_r$$

is true.

---

1) Instead of

$M(f_1)(\Delta) \geq 0, ..., M(f_k)(\Delta) \geq 0; M(f_1)(\Delta) < 0$ or ... or $M(f_k)(\Delta) < 0$

we have written $(\forall i)\ M(f_i)(\Delta) \geq 0$, $(\exists i)\ M(f_i)(\Delta) < 0$ respectively.

Proof. Let first $(x_1, ..., x_n)$ be a solution of system (2.1). Then according to (2.2) for every $r \in \mathbb{N}$ we have $(x_1, ..., x_n) \in F_r$ which implies

$$(x_1, ..., x_n) \in \bigcap_{r \in \mathbb{N}} F_r$$

So, we have proved the inclusion $S \subseteq \bigcap_{r\in\mathbb{N}} F_r$. Suppose now that $(x_1, ..., x_n) \notin S$. Then at least one of the inequalities

$$f_j(x_1, ..., x_n) < 0$$

must be true. Since f_j is a continuous function there exist a negative number p and an n-dimensional segment $\Delta \subseteq D$ such that $(x_1, ..., x_n)$ is one of its interior points and additionally the inequality

(*1) $$f_j(x_1, ..., x_n) < p$$

is true whenever $(x_1, ..., x_n) \in \Delta$. Further according to axiom (0.2) there exists $\delta > 0$ such that for all n-dimensional segments $\Delta' \subseteq \Delta$, with $(x_1, ..., x_n) \in \Delta'$, $diam\Delta' \leq \delta$ the following inequality

(*2) $$M(f_j)(\Delta') - m(f_j)(\Delta') < -p/2$$

is true. Using (*1), (*2) we conclude

$$\begin{aligned} M(f_j)(\Delta') &< m(f_j)(\Delta') - p/2 \\ &\leq p - p/2 \quad \text{(By axiom (0.1))} \\ &< 0 \end{aligned}$$

Based on that we infer that there exists a natural number r such that for any

$$P_r \in \mathcal{D}_r(D)$$

containing the point $(x_1, ..., x_n)$ the inequality

$$M(f_j)(P_r) < 0$$

holds. Consequently for such an r the point $(x_1, ..., x_n)$ cannot be an element of the set F_r. Thus we have proved the implication

$$(x_1, ..., x_n) \notin S \Rightarrow (x_1, ..., x_n) \notin \bigcap_{r\in\mathbb{N}} F_r$$

which completes the proof.

Obviously Theorem 2.1 suggests an idea how to solve system (2.1), briefly said how to find F_r step-by-step. To improve such a procedure we can define the set F_{r+1} as a subset of the set F_r. This idea is used in the following solving procedure for a given system (2.1):

(2.3) *Solving procedure depends on a cell-decomposition $\mathcal{D}(D)$. If we want it to be a cell-tree then we can choose it in advance. In opposite case we determine*

$\mathcal{D}_r(D)$ during the solving procedure. Further, step-by-step we form a sequence $\langle F'_r \rangle$ whose each member F'_r is the union of some feasible products[2] *$P_r \in \mathcal{D}_r(D)$. This sequence is defined inductively as follows*

1^0 $F'_0 = D$ if D is feasible, otherwise $F'_0 = \emptyset$.
2^0 For any $r \in \mathbb{N}$

F'_{r+1} = *The union of all products $P_{r+1} \in \mathcal{D}_{r+1}(D)$ such that P_{r+1} is feasible and $P_{r+1} \subseteq F'_r$.*

If $\mathcal{D}(D)$ is a cell-tree then condition $P_{r+1} \subseteq F'_r$ is satisfied, according to the definition of a cell-tree. Otherwise, we should define $\mathcal{D}_{r+1}[a_1, b_1]$, ..., $\mathcal{D}_{r+1}[a_n, b_n]$ in the $(r+1)^{-th}$ step so that the condition[3] *$P_{r+1} \subseteq F'_r$ be satisfied.*

If for some $r \in \mathbb{N}$ we have the equality $F'_r = \emptyset$ then the procedure halts and S is $\emptyset$. Otherwise, the sets F'_r when r is getting greater and greater give better and better approximations of the set S.

Notice that the sequences $\langle F_r \rangle$, $\langle F'_r \rangle$ may differ but nonetheless the equality

$$\cap_{r\in\mathbb{N}} F_r = \cap_{r\in\mathbb{N}} F'_r$$

always holds. Besides that the sequence $\langle F'_r \rangle$ is monotone, for the inclusions

$$F'_0 \supseteq ... \supseteq F'_r \supseteq F'_{r+1} \supseteq ...$$

are satisfied. Generally about the nature of the procedure one may say the following:

(2.4) *Using the fact that non feasible cells cannot contain ony solution we actually reject step-by-step various solution-free "peaces" of the given domain D. Additionally, the non-feasibility criterion is so fine that every point $(x_1, ..., x_n) \in D$, which is not a solution, will be rejected in some step r. Accordingly if system (2.1) has no solutions then in some step r all products $P_r \subseteq F'_r$ will be non-feasible, which implies the conclusion $S = \emptyset$.*

As we shall see later for almost all aplications m-M calculus certain solving procedure like (2.3) will be used and consequently something like (2.4) will be valid. About system (2.1) we also add the following. For some products P_r it may happen that all inequalities

$$m(f_1)(P_r) \geq 0, ..., m(f_k)(P_r) \geq 0$$

are satisfied. Obviously such products must be subsets of the set S. Consequently we have the following definition.

Definition 2.2. *An n-dimensional segment $\Delta \subseteq D$ is a solutional segment in the sense of system (2.1) if the condition $(\forall i)\, m(f_i)(\Delta) \geq 0$ is fulfilled.*

[2] According to Definition 1.2, 2^0 elements of $\mathcal{D}_r(D)$ are some Cartesian products.
[3] For instance, $n = 1$, $a_1 = 1$, $b_1 = 1$, $F_r = [3,4] \cup [4,5] \cup [5,6]$, i.e. $F_r = [3,6]$. Then in the $r+1^{th}$ step instead defining $\mathcal{D}_7[a_1, b_1]$ we can define $\mathcal{D}_7[3,6]$ (for example dividing the segment $[3,6]$ into 100 equal subsegments).

Besides this definition, the so-called indetermined segments are defined by the following conditions

(2.5) $$(\forall i)\, M(f_i)(\Delta) \geq 0, \quad (\exists i)\, m(f_i)(\Delta) < 0$$

In other words, a segment $\Delta \subseteq D$ is indetermined if and only if Δ is a feasible but not a solutional segment. Using the notions of solutional and indetermined products the solving procedure (2.3) can be profoundly improved as follows[4)]:

(2.6) *Step-by-step we form sequences $\langle S_r \rangle$, $\langle U_r \rangle$ whose members S_r, U_r are unions of some solutional, indetermined products P_r respectively. Their inductive definitions reads:*

1^0 $S_0 = D$ *if D is a solutional product, otherwise* $S_0 = \emptyset$
$U_0 = D$ *if D is an indetermined product, otherwise* $U_0 = \emptyset$

2^0 *For any* $r \in \mathbb{N}$
$S_{r+1} = S_r \cup$ *The union of all solutional products* $P_{r+1} \subseteq P_r$
$U_{r+1} =$ *The union of all indetermined products* $P_{r+1} \subseteq P_r$

If for some $r \in \mathbb{N}$ we obtain the equality $S_r \cup U_r = \emptyset$ then the procedure halts and the equality $S = \emptyset$ is true. Similarly, if for some $r \in \mathbb{N}$ $U_r = \emptyset$ then the procedure halts too and the equality $S = S_r$ is true.
Otherwise, i.e. if for every $r \in \mathbb{N}$ both relations $S_r \cup U_r \neq \emptyset$, $U_r \neq \emptyset$ are fulfilled, the sets $S_r \cup U_r$ when r is getting greater and greater give better and better approximations of the set S.

Notice that the sequences $\langle S_r \rangle$, $\langle U_r \rangle$ have the following properties:

$$S_0 \subseteq S_1 \subseteq \dots \subseteq S_r \subseteq S_{r+1} \dots, \quad U_0 \supseteq U_1 \supseteq \dots \supseteq U_r \supseteq U_{r+1} \dots,$$

$$F'_0 = S_0 \cup U_0, \dots, F'_r = S_r \cup U_r, \dots$$

About the solving procedure (2.6) we add the following. Bearing in mind Remark 1.4 one can extend this procedure to the case of system (2.1) when certain of functions f_i are defined by some terms from the set (1.11).

2. Let now again $D = [a_1, b_1] \times \dots \times [a_n, b_n]$ be a given n-dimensional segment and let $f_1, \dots, f_k : D \to \mathbb{R}$ be given m-M functions. In connection with them we consider the following system of equations in $x_1, \dots, x_n$

(2.7) $$f_1(x_1, \dots, x_n) = 0, \dots, f_k(x_1, \dots, x_n) = 0$$

supposing that $a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n$. Notice that this system can be reduced to the following system of inequalities

$$f_1 \geq 0, -f_1 \geq 0, \quad \dots, \quad f_k \geq 0, -f_k \geq 0$$

$$a_1 \leq x_1 \leq b_1, \quad \dots, \quad a_n \leq x_n \leq b_n$$

[4)] The details about $\mathcal{D}(D)$, stated in (2.3), are supposed.

Accordingly we may directly apply the procedure described above. The only additional thing we want to emphasize is that using Definition 2.1 we get the following feasibility-definition for system (2.7)

Definition 2.3. *An n-dimensional segment $\Delta \subseteq D$ is feasible set in the sense of system (2.7) if the condition*

(2.8) $$(\forall i)\, (m\,(f_i)\,(\Delta) \leq 0 \leq M\,(f_i)\,(\Delta))$$

is satisfied.

According to this some product P_r will be rejected in solving procedure if P_r satisfies the following condition

(2.9) $$(\exists i)\, (m\,(f_i)\,(P_r) > 0 \text{ or } M\,(f_i)\,(P_r) < 0)$$

which is the negation of (2.8).

Let now $f : D \to \mathbb{C}$ ($dim\ D = 2$) be a given complex function and suppose that at least one effective formula for $m\,(|f|)$ is known (see (1.21)). Then solving the equation $f(z) = 0$ in $z \in D$ may be treated as solving the real equation $|f(z)| = 0$.Now the definition of feasible segment $\Delta \subseteq D$ reads:

(2.10) *A segment $\Delta \subseteq D$ is feasible in the sense of equation $|f(z)| = 0$ if the condition $m\,(|f|)\,(\Delta) \leq 0$ holds.*

An important particular case is provided when f is a polynomial function determined, say, by

$$f(z) = a_n z^n + \dots + a_0 \qquad (a_n \neq 0)$$

where $a_n, \dots, a_0$ are given complex numbers. Then a domain $D = [-r, r] \times [-r, r]$ which contains all the solutions of the equation $f(z) = 0$ can be effectively found. For example, by Caushy's formula for the number r we can take

(2.11) $$r = 1 + \max_{0 \leq i \leq n-1} (|a_i|/|a_n|)$$

Notice also that previous ideas about solving complex equations $f(z) = 0$ can be extended to the case of solving systems of such equations

$$f_1(z_1, \dots, z_n) = 0, \dots, f_k(z_1, \dots, z_n) = 0$$

Then, roughly speaking, the corresponding criterium for feasibility reads

$$(\forall i)\, m\,(|f_i|)\,(P_r) \leq 0$$

3. Now, we give various examples. We emphasize that generally $fis(r)$ will denote the number of all feasible products of r-cells.

Example 2.1. *Equation $sinx = 1/x$, $\quad x \in [1, 20]$.*

Let $[\alpha, \beta] \subseteq [1, 20]$ be any segment. Then according to Definition 1.1. (ix) for the function $f(x) = \sin x - 1/x$ one m-M pair is defined by

$$m\,(f)\,[\alpha, \beta] = \alpha + \sin \alpha - \beta - 1/\alpha, \quad M\,(f)\,[\alpha, \beta] = \beta + \sin \beta - \alpha - 1/\beta$$

By the procedure (2.3), using the feasibility definition of the type 2.2 and diadic tree, the number $fis(r)$ of all feasible r-cells step-by-step up to $r = 25$ is given in the following list (its elements have the form (step r,fis(r)).

(1,1), (2,2), (3,4), (4,8), (5,15), (6,16), (7,16), (8,15), (9,16), (10,14), (11,14), (12,16), (13,16), (14,15), (15,15), (16,15) , (17,17), (18,16), (19,15), (20,15), (21,15), (22,15), (23,15) , (24,15), (25,15)

As we see starting with the step 5 the number of all feasible cells $fis(r)$ is about 16. Consequently, according to (2.4), in these steps we should test about $16 \cdot 2$ (i.e. 32) cells only. For instance, exactly said, in the 20^{th} step there are all together 2^{20} cells,but we should test only 30 of them. In the step 25 we obtain the following numerical result:

The given equation has 7 solutions described as follows

$$1.11415595 \le x_1 \le 1.11415821$$
$$2.77260345 \le x_2 \le 2.77260572$$
$$6.4391157 \le x_3 \le 6.4391191$$
$$9.31724286 \le x_4 \le 9.31724399$$
$$12.6455307 \le x_5 \le 12.6455341$$
$$15.6439972 \le x_6 \le 15.6439983$$
$$18.9024819 \le x_7 \le 18.9024853$$

Example 2.2. *Complex equation in* z $(= x + iy)$

$$z^8 + (A_7 + iB_7)\, z^7 + \dots + (A_0 + iB_0) = 0$$

where $A_7, B_7, \dots, A_0, B_0$ *are given real numbers.*

All solutions lie in the domain $[-r, r] \times [-r, r]$ where r is defined by (2.11). Using the procedure (2.3), Definition of type(2.10) and diadic trees, several equations are solved up to 25^{th} step. In all of them the coefficienst were chosen at random. It is interesting, the numbers $fis(r)$, when $r \ge 6$ are pretty small. Namely, in the 25^{th} step this number is always less then 15. We give concrete numerical results in the case when coefficients A_j, B_j are determined as follows

$$A_7 = -0.628871968 \qquad B_7 = -0.90620273$$
$$A_6 = 0.655487601 \qquad B_6 = 0.109498452$$
$$A_5 = 0.794467662 \qquad B_5 = 0.145832495$$
$$A_4 = 0.677786328 \qquad B_4 = 0.862459254$$
$$A_3 = -0.623235982 \qquad B_3 = 0.945879881$$
$$A_2 = 0.552867495 \qquad B_2 = -0.164039785$$
$$A_1 = 0.658555102 \qquad B_1 = 0.618662189$$

$$A_0 = 0.934256145 \qquad B_0 = 0.147878684$$

The solutions $x_i + iy_j$ $(j = 1, \dots, 8)$ are described as follows

$$-0.340724289 \le x_1 \le -0.34072414, \qquad -0.793053508 \le y_1 \le -0.793053359$$
$$-0.897440463 \le x_2 \le -0.897440314, \qquad -0.308772177 \le y_2 \le -0.308772027$$
$$0.385927558 \le x_3 \le 0.385927707, \qquad -0.954408497 \le y_3 \le -0.954408199$$
$$1.11732647 \le x_4 \le 1.11732662, \qquad -0.46066165 \le y_4 \le -0.460661501$$
$$-0.310650319 \le x_5 \le -0.31065017, \qquad 0.521920323 \le y_5 \le 0.521920621$$
$$-0.707707405 \le x_6 \le -0.707707107, \qquad 0.786857605 \le y_6 \le 0.786857754$$
$$0.643312931 \le x_7 \le 0.64331308, \qquad 0.49212873 \le y_7 \le 0.492128879$$
$$0.738826841 \le x_8 \le 0.73882699, \qquad 1.62219122 \le y_8 \le 1.62219137$$

Example 2.3. *Complex equation in* $z = x + iy$

$$e^z = z$$

In the domain $[-20, 20] \times [-20, 20]$ this equation has 6 solutions $x_j + iy_j$ $(j = 1, \dots, 6)$ desribed as follows

$$2.65319109 \le x_1 \le 2.65319228\,, -13.94920826 \le y_1 \le -13.94920731$$
$$2.06227660 \le x_2 \le 2.06227899, \quad -7.58863215 \le y_2 \le -7.58863020$$
$$0.31813025 \le x_3 \le 0.31813264, \quad -1.33723736 \le y_3 \le -1.33723497$$

$$x_4 + iy_4 = x_3 - iy_3$$
$$x_5 + iy_5 = x_2 - iy_2$$
$$x_6 + iy_6 = x_1 - iy_1$$

The calculations were done up to the 25^{th} step (bisection way). Starting with the 6^{th} step the number $fis(r)$ was about 16. For instance:

$$fis(24) = 15, \quad fis(25) = 16$$

Example 2.4. *We consider the system in* $(x, y, z) \in D \subset \mathbb{R}^3$

$$e^x + x + \sin y + \cos z = p$$
$$x^3 + e^{\sin y} - z - e^z = q$$
$$\sin(x - z) + (x + y)^5 - x - y - z = r$$

where p, q, r *are some given real numbers.*

Notice that in all cases stated below again diadic trees are used.

C a s e 1: $p = 2$, $q = 0$, $r = 0$, $D = [1, 2] \times [-2, 1] \times [-3, 2]$.

There is exactly one solution $(x, y, z) = (0, 0, 0)$. Starting with the 6^{th} step the number $fis(r)$ was between 40 and 50. In the 24^{th} step we obtained the following result

$$-0.0000152587891 \le x \le 0.0000247955322$$
$$-0.0000247955322 \le y \le 0.0000324249268$$
$$-0.00000762939453 \le z \le 0.0000114440918$$

C a s e 2: $p = 2$, $q = 0$, $r = 0$, $D = [-5, 5] \times [1, 5]$.
Step-by-step the number $fis(r)$ is

$$1,\ 8,\ 21,\ 32,\ 24,\ 0$$

Accordingly, we conclude that the system has no solutions.

Remark 2.1. *This example illustrates one of the key features of the m-M calculus generally:*

(2.12) *If some problem[5] has no solution in a given domain D then there exists a step k such that $fis(k) = 0$.*

In other words, the non-existence of solutions can be positively established in some step k.

C a s e 3: $p = 3, q = -2, r = 5, D = [1, 10] \times [1, 10] \times [-10, 10]$.
The system has no solutions since $fis(3) = 0$

4. Now we are going to study the feasibility problem, in other words the convergence rate problem. First, we introduce some notions.

Let $D = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$ be a given domain and let $\mathcal{D}(D)$ be some cell-decomposition (see Definition 1.2). For a fixed $r \in \mathbb{N}$ consider any two elements *Cell1, Cell2* of the set $\mathcal{D}_r(D)$. In connection with them we introduce the following definition

(2.13) *The elements Cell1, Cell2 are called neighbouring if they have at least one joint vertex[6]*

Next using (2.13) we have a new definition:

(2.14) *A subset $A_r \subseteq \mathcal{D}_r(D)$ is called a cell-connected set if for any two elements $\Delta', \Delta'' \in A_r$ there exist elements $\Delta_1, ..., \Delta_k \in A_r$ such that $\Delta' = \Delta$, $\Delta'' = \Delta_k$ and for each $i \in \{1, ..., k-1\}$ the elements Δ_i, Δ_{i+1} are neighbouring.*

Let now $f(x) = 0$ with $x \in D$ be an equation in x, where f is a given m-M function and $D = [a, b]$ given domain. Suppose that $c \in [a, b]$ is a solution. If $\mathcal{D}[a, b]$ is any cell-decomposition then in each step r some of its r-cells must be feasible. For

[5] Above we have a case of a system of equations. But as we shall see further the same fact holds for any problem treated by m-M calculus

[6] For instance, in case n=1 the cells $[\alpha, \beta]$ and $[\gamma, \delta]$ are neighbouring if and only if the condition: $\beta = \gamma$ or $\delta = \alpha$ is satisffied

instance,at least the cell[7] $C_r(c)$ must be such one. Denote by $Fis(c, r)$ the maximal set having the following properties

(2.15) (i) *Each element of $Fis(c, r)$ is a feasible r-cell.*

(ii) $C_r(c) \in Fis(c, r)$

(iii) *$Fis(c, r)$ is cell-connected set.*

Intuitively, the union of all elements of the set $Fis(c, r)$ approximately determines the solution c. Accordingly the error in the r-th step is defined as follows

$$E_r = diam\left(\bigcup Fis(c, r)\right) \tag{2.16}$$

Next denote by $fis(c, r)$ the cardinal number of the set $Fis(c, r)$. In m-M calculus the rate of convergence directly depends on the magnitude of these two numbers. Of course the ideal case is when we can use ordinary $\min(f)$, $\max(f)$ for $m(f)$, $M(f)$ respectively. Then, for instance, if c is an isolated solution of the equation $f(x) = 0$ the number $fis(c, r)$ will be 1 or[8] 2. In the sequel the maximum of all $diam\ \delta$ with $\delta \in \mathcal{D}_r[a, b]$ will be denoted by $d(r)$.

Remark 2.2. *Concerning the error formula we shall often prove a certain inequality of the form*

$$E_r \le L\,(d(r))^\sigma \quad (\text{for every } r \ge r_0) \tag{*}$$

In this inequality L, σ are some positive reals and r_0 is some member of $\mathbb{N}$. If $\sigma = 1$ and $\mathcal{D}[a, b]$ is a tree such that the diameter of any element of the set $T_r[a, b]$ is $(b - a)/d^r$ where $d > 1$ is a fixed real number, then the inequality (*) becomes

$$E_r \le L(b - a)/d^r$$

which means that $fis(c, r)$ is bounded by L. Such a case is close to the case when $m(f)$, $M(f)$ are equal to $\min(f)$, $\max(f)$ respectively.

Theorem 2.2. *Let $f(x) = 0$, $x \in [a, b] \subset \mathbb{R}$ be an equation in x, where $f : [a, b] \to \mathbb{R}$ is a given m-M function for which $\langle m(f), M(f)\rangle$ is a σ-Lipschitz m-M pair. Let $c \in [a, b]$ be a solution of the equation $f(x) = 0$ such that the following condition is satisfied.*

(2.17) *In some neighbourhood $\Delta \ni c$ the function f has continuous derivative f' and $f'(c) \ne 0$.*

Then if we use the solving procedure (2.3) we can see that there exist a positive constant L and $r_0 \in \mathbb{N}$ such that for every $r \ge r_0$ the inequality

$$E_r \le L \cdot (d(r))^\sigma$$

holds.

[7] Recall that $C_r(c)$ contains c as an element

[8] For further details see Example 2.5.

Proof. Let $|f'(c)| = p$. In virtue of (2.17) and Theorem 2.1 there exists $r_0 \in \mathbb{N}$ such that for every $r \geq r_0$ the following conditions

(i) $\bigcup Fis(c,r) \subseteq [a,b]$

(ii) To each element $\Delta \in Fis(c,r)$ we may apply the inequality from Definition 1.8, i.e. the inequality

$$|M(f)(\Delta) - m(f)(\Delta)| \leq K(diam\Delta)^\sigma \qquad \text{(K is some positive constant)}$$

are fulfilled. Let δ be any element of the set $Fis(c,r)$. Then for $r \geq r_0$ we have the following implicational argument:

δ is feasible

$\Rightarrow m(f)(\delta) \leq 0 \leq M(f)(\delta)$

$\Rightarrow f(s) - K(diam\ \delta)^\sigma \leq 0 \leq f(s) + D(diam\ \delta)^\sigma$, where s is any element of the δ. This follows from Remark 1.6.

$\Rightarrow |f(s)| \leq K(diam\ )^\sigma$

$\Rightarrow |f(c) + (s-c)f'(\xi)| \leq K(diam\ \delta)^\sigma$ (for some $\xi \in \delta$)

$\Rightarrow |s - c| \leq \frac{2K}{p}(diam\ \delta)^\sigma$

from which it is easy to prove an inequality of the type

$$diam\,(\bigcup Fis(c,r)) \leq L \cdot (d_r)^\sigma \qquad (L \text{ is a constant})$$

i.e. the inequality (*).

Theorem 2.3. *Let $f(x) = 0$, $x \in [a,b]$ be an equation in x, where $f : [a,b] \to \mathbb{R}$ is a given m-M function and let k be a non-negative integer. Suppose that $\langle m(f), M(f)\rangle$ is a k-Taylor m-M pair and that for some $c \in (a,b)$ the following conditions are satisfied*

(2.18) In some neighbourhood $\Delta \ni c$ function f has continuous derivative $f^{(k+1)}(x)$ and $f(c) = 0,\ f'(c) = 0,\ \ldots,\ f^{(k)}(c) = 0,\ f^{(k+1)}(c) \neq 0.$

Then if we use the solving procedure (2.3) we can see that there exist a positive constant L and $r_0 \in \mathbb{N}$ such that for every $r \geq r_0$ the inequality

$$E_r \leq L \cdot d(r) \tag{2.19}$$

holds.

Proof. In virtue of (2.18) and Theorem 2.1 there exists $r_0' \in \mathbb{N}$ such that for every $r \geq r_0'$ the condition: $\bigcup Fis(c,r) \subseteq [a,b]$ is fulfilled. According to (1.18) there exist $r_0'' \geq r_0'$ and a positive constant K such that for all $\delta \in Fis(c,r)$ with $r \geq r_0''$ the inequality

$$B\left(\left|f^{(k+1)}\right|\right)(\delta) \leq K \tag{$*_1$}$$

holds. Next, let $r \geq r_0''$ and let δ be any element of the set $Fis(c,r)$. Then using (1.19) and the inequality $m(f)(\delta) \leq 0 \leq M(f)(\delta)$ we obtain the following inequality

$$|f(\gamma)| \leq \sum_{i=1}^{k} \frac{|f^{(i)}(\gamma)|}{i!}\rho^i + \frac{\rho^{k+1}}{(k+1)!}B\left(\left|f^{(k+1)}\right|\right)(\delta) \tag{$*_2$}$$

where $\rho = diam\ \delta$ and γ is any element of δ (see Remark 1.5). By Taylor's formula we have the equalities

$$\begin{aligned} f(\gamma) &= f(c) + \ldots + \frac{(\gamma-c)^k}{k!}f^{(k)}(c) + \frac{(\gamma-c)^{k+1}}{(k+1)!}f^{(k+1)}(\xi_0) \\ f'(\gamma) &= f'(c) + \ldots + \frac{(\gamma-c)^{k-1}}{(k-1)!}f^{(k)}(c) + \frac{(\gamma-c)^k}{k!}f^{(k+1)}(\xi_1) \\ &\ldots \\ f^{(k)}(\gamma) &= f^{(k)}(c) + (\gamma-c)f^{(k+1)}(\xi_k) \end{aligned} \tag{$*_3$}$$

where $\xi_0, \ldots, \xi_k$ are some numbers between c and γ. Bearing in mind $(*_1)$, $(*_3)$ and (2.18) inequality $(*_2)$ implies the following one

$$q^{k+1} \leq \sum_{i=1}^{k} A_i \binom{k+1}{i} q^{k+1-i} + A_{k+1} \qquad (q > 0) \tag{$*_4$}$$

where the following denotations

$$q = \frac{\gamma - c}{\rho}, \qquad A_i = \left|\frac{f^{(k+1)}(\xi_i)}{f^{(k+1)}(\xi_0)}\right|, \qquad A_{k+1} = \frac{K}{|f^{(k+1)}(\xi_0)|}$$

are employed. Obviously $A_{k+1} \geq 1$. If $r \to \infty$ then $diam\ \delta \to 0$ and according to (2.18) and (0.2) the numbers $A_1, \ldots, A_k$ tend to 1, while A_{k+1} tends to $K/\left|f^{(k+1)}(c)\right|$, which is greater or equal to 1. Putting $A_1 = \ldots = A_k = 1$, $A_{k+1} = K/\left|f^{(k+1)}(c)\right|$ in $(*_4)$ we obtain the following q-inequality

$$2q^{k+1} \leq (1+q)^{k+1} + P \qquad \left(P := -1 + \frac{K}{|f^{(k+1)}(c)|}, \quad P > 0\right) \tag{$*_5$}$$

It can be easily proved that the corresponding equation has exactly one positive solution, say λ. Consequently, $(*_5)$ implies the inequality

$$0 < q \leq \lambda$$

Let $\varepsilon > 0$ be a positive number. Since $(*_5)$ is the limit-inequality of the inequality $(*_4)$ when $r \to \infty$ there exists some $r_0 \geq r_0''$ such that for every $r \geq r_0$ the inequality $(*_4)$ implies the following inequality

$$q < \lambda + \varepsilon$$

Hencefrom we conclude (2.19), by which it is easy to complete the proof.

Remark 2.3. *It is not difficult to see that Theorem 2.3 can be in a clear way transfer to the case of some complex equation*

$$f(z) = 0, \qquad \textit{where } z \in D \subset \mathbb{C}$$

Then of course for $\langle m(f), M(f)\rangle$ we should apply the formulas (1.21).

Theorem 2.3, among others, says that if a solution $c \in [a, b]$ satisfies the conditions

$$f(c) = 0, ..., f^{(k)}(c) = 0, \; f^{(k+1)}(c) \neq 0$$

i.e. if c is a $(k+1)$-fold solution, then by using k-Taylor m-M pair in solving procedure of the type (2.3) one achieves that the sequence $\langle fis(c, r)\rangle$ becomes bounded. Of course in order to achieve such a thing we could use any k'-Taylor m-M pair providing that $k' \geq k$.

Now we pass to the convergence rate problem of the system of equations (2.7). If $(c_1, ..., c_n)$ is any of its solutions then the set $Fis(c_1, ..., c_n, r)$ is defined by a definition of the type (2.15). Accordingly $fis(c_1, ..., c_n, r)$ is a cardinal number of this set. The error, like (2.16) is defined by

$$E_r = diam\left(\bigcup Fis(c_1, ..., c_n, r)\right)$$

Remark 2.4 *(Continuation of Remark 2.2). In this case we are also interested in some inequality of the form (*), where now $d(r)$ denotes the maximum of all diam δ with $\delta \in \mathcal{D}_r[a_1, b_1] \times ... \times \mathcal{D}_r[a_n, b_n]$. Let $\delta = \delta_1 \times ... \times \delta_n$. If $\sigma = 1$ and $\mathcal{D}[D]$ is a cell-tree such that*

(2.20) $$diam\ \delta_i = \frac{b_i - a_i}{d_i^r} \qquad (i = 1, 2, ..., n)$$

where $d_i > 1$ are some fixed real numbers, then inequality (*) becomes

$$E_r \leq L \cdot \left(\sum_{i=1}^{n} \frac{(b_i - a_i)^2}{d_i^{2r}}\right)^{1/2}$$

which means that $fis(c_1, ..., c_n, r)$ is bounded by L.

Theorem 2.4. *Let*

(2.21) $$f_1(x_1, ..., x_n) = 0, ..., f_n(x_1, ..., x_n) = 0 \quad (x_1, ..., x_n) \in D$$

be a system of equations where $f_1, ..., f_n : D \to \mathbb{R}$ are given m-M functions for which $\langle m(f_i), M(f_i)\rangle$ are some σ_i-Lipschitz's m-M pairs. Let $c_1, ..., c_n) \in D$ be a solution and also let the following condition be fulfulled

(*) *In some neighbourhood $\Delta \ni (c_1, ..., c_n)$ the functions $f_1, ..., f_n$ have continuous first order partial derivatives and their Jacobian does not vanish at the point $(c_1, ..., c_n)$*

Then if we use the solving procedure (2.3) we can see that there exist positive constants L, σ and $r_0 \in \mathbb{N}$ such that for every $r \geq r_0$ the inequality

(**) $$E_r \leq L\,(d(r))^{\sigma}$$

holds.

Proof. Let

$$\mathcal{J} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(x_1, ..., x_n) & \cdots & \frac{\partial f_n}{\partial x_1}(x_1, ..., x_n) \\ \frac{\partial f_1}{\partial x_n}(x_1, ..., x_n) & \cdots & \frac{\partial f_n}{\partial x_n}(x_1, ..., x_n) \end{vmatrix}$$

be the Jacobian at any point $(x_1, .., x_n) \in \Delta$. In virtue of (*) there exists a neighbourhood $\Delta' \ni (c_1, ..., c_n)$, $\Delta' \subseteq \Delta$ such that the determinant[9)]

$$\mathcal{J}^* = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(x_{11}, ..., x_{n1}) & \cdots & \frac{\partial f_n}{\partial x_1}(x_{11}, ..., x_{n1}) \\ \cdots\cdots & \cdots & \cdots\cdots \\ \frac{\partial f_1}{\partial x_n}(x_{1n}, ..., x_{nn}) & \cdots & \frac{\partial f_n}{\partial x_n}(x_{1n}, ..., x_{nn}) \end{vmatrix}$$

does not vanish whenever

$$(x_{11}, ..., x_{n1}) \in \Delta', ..., (x_{1n}, ..., x_{nn}) \in \Delta'$$

and moreover $|\mathcal{J}^*| \geq p/2$ where p is the value of the Jacobian at the point $(c_1, ..., c_n)$. Hence we conclude that there exists a neighbourhood $\Delta'' \ni (c_1, ..., c_n)$, $\Delta'' \subseteq \Delta'$ such that the point $(c_1, ..., c_n)$ is the unique solution of system (2.21) in the domain Δ''. Next, Theorem 2.1 implies that there exists a natural number r_0 such that $Fis(c_1, ..., c_n, r)$ is a subset of the set Δ'' whenever $r \geq r_0$. Additionally suppose that r_0 has been chosen so that for $r \geq r_0$ the inequality $diam\ \delta \leq 1$ holds for any $\delta \in \mathcal{D}_r(D)$.

Let now δ be any element of the set $Fis(c_1, ..., c_n, r)$. Then for $r \geq r_0$ we have the following implication argument

δ is a feasible product

$\Rightarrow$ $(\forall i)\,(m(f_i)(\delta) \leq 0 \leq M(f_i)(\delta))$

$\Rightarrow$ $(\forall i)\,(f_i(s_1, ..., s_n) - K\,(diam\,\delta)^{\sigma_i} \leq 0 \leq f_i(s_1, ..., s_n)i + K\,(diam\,\delta)^{\sigma_i})$ where K is some positive constant and $(s_1, ..., s_n)$ is any point of δ. That follows from Remark 1.6.

$\Rightarrow$ $(\forall i)\,|f_i(s_1, ..., s_n)| \leq K\,(diam\ \delta)^{\sigma_i}$

$\Rightarrow$ $(\forall i)\,|f_i(s_1, ..., s_n)| \leq K\,(diam\ \delta)^{\sigma}$ where $\sigma = \min(\sigma_1, ..., \sigma_n)$, providing that $diam\ \delta \leq 1$.

$\Rightarrow$ $(\forall i)\left|f_i(c_1, ..., c_n) + \sum_{j=1}^{n}(s_j - c_j) \cdot \frac{\partial f_i}{\partial x_j}(x_{1i}, ..., x_{ni})\right| \leq K\,(diam\,\delta)^{\sigma}$ by Lagrange's theorem; for some $x_{1i}, ..., x_{ni} \in \delta$, where $i = 1, ..., n$.

9) It follows from the fact that $\mathcal{J}$ is a polynomial in $\partial f_i / \partial x_j$.

$\Rightarrow$ $(\forall i)\sum_{i=1}^{n}(s_j - c_j)\frac{\partial f_i}{\partial x_j}(x_{1i},...,x_{ni}) = \theta_i K\,(diam\ \delta)^{\sigma}$, where $\theta_i \in [-1,1]$ are some real numbers.

$\Rightarrow$ $(\forall j)\,(s_j - c_j)\cdot\mathcal{J}^* = P_j\cdot(diam\ \delta)^{\sigma}$ by Cramer's theorem, where $P_1,...,P_n$ are the corresponding determinants.

$\Rightarrow$ $(\forall j)\,|s_j - c_j| \leq 2p^{-1}|P_j|\,(diam\ )^{\sigma}$, for: $|\mathcal{J}^*| \geq p/2$

$\Rightarrow$ $(\forall j)\,|s_j - c_j| \leq L'\,(diam\ \delta)^{\sigma}$, where L' is some constant.

$\Rightarrow$ $\sum_{j=1}^{n}(s_j - c_j)^2 \leq L'^2\,(diam\ \delta)^{2\sigma}$

$\Rightarrow$ $diam\,(\bigcup Fis(c_1,...,c_n,r)) \leq L\,(diam\ \delta)^{\sigma}$, where $L = 2L'$

$\Rightarrow$ $diam\,(\bigcup Fis(c_1,...,c_n,r)) \leq L\,(dr)^{\sigma}$

$\Rightarrow$ (**)

Now we will consider system (2.7) which is more general than system (2.21). Concerning that system we assume:

(2.22) 1^0 $f_1,...,f_k : D \to \mathbb{R}$ are given m-M functions and $\langle m(f_i), M(f_i)\rangle$ are some 1-Lipschitz's m-M pairs of them.

2^0 In D the functions $f_1,...,f_k$ have all the first order partial derivatives, and these derivatives are m-M functions.

3^0 The use of solving procedure (2.3).

4^0 $c = (c_1,...,c_n) \in D$ is an isolated solution of the system.

Notice that the condition: $\langle m(f_i), M(f_i)\rangle$ are 1-Lipschitz's m-M pairs, is not a strong restriction (see[10] Theorem 1.3). The main practical problem which can appear is

(σ) The sequence $fis(c_1,...,c_n)$ is not bounded

Denote by $\mathcal{J}_1,...,\mathcal{J}_{\binom{k}{n}}$ all minors of order n of the Jacobi matrix

$$\left\|\frac{\partial f_i}{\partial x_j}(c_1,...,c_n)\right\|$$

Bearing in mind Definition 2.3 and the given proof of Theorem 2.4 one necessary and sufficient condition for (δ) is

All minors $\mathcal{J}_1,...,\mathcal{J}_{\binom{k}{n}}$ vanish at the point c

Accordingly:

(2.23) *If we have case (σ) we may replace system (2.7) by this one*

(2.7), plus equations: $\mathcal{J}_1 = 0,\ ...,\ \mathcal{J}_{\binom{k}{n}} = 0$

[10] Any Taylor's m-M pair is also 1-Lipshitz's m-M pair.

We emphasize that (2.23) is the main idea used to diminish the number of feasible cells. Of course this idea can be used several times[11]. As an illustration we give two small examples.

Example 2.5 *Consider the system*

$$f(x,y) = 0, g(x,y) = 0 \quad (where\ f(x,y) = y - x^2, g(x,y) = y)$$

in $(x,y) \in D = [0,1] \times [0,1]$.
If $\Delta = [\alpha,\beta] \times [\gamma,\delta]$ is any subsegment of D then applying Definition 1.1 we have the following formulas

$$m(f)(\Delta) = \gamma - \beta^2, M(f)(\Delta) = \delta - \alpha^2; \quad m(g)(\Delta) = \gamma, M(g)(\Delta) = \delta$$

These formulas are of the type (1.1). However, despite of that fact, the members of the sequence[12] $fis(0,0,r)$ are in turn

1, 2, 3, 3, 5, 6, 9, 12, 17, 23, 33
46, 65, 91, 129, 182, 257, 363,...

It is not difficult to prove that $fis(0,0,r)$ is the number of all i-solutions of the inequality

$$i^2 \leq 2^r \quad (\text{assuming } i = 0,1,2,...)$$

Consequently $fis(0,0,r)$ is not bound. Can it be compatible with the fact that $m(f)$, $M(f)$, $m(g)$, $M(g)$ are $\min(f)$, $\max(f)$, $\min(g)$, $\max(g)$ respectively? The main point is the following:

Generally, in the equivalence

$$\begin{array}{c} For\ some\ x,y \in \Delta, \\ f(x,y) = 0, g(x,y) = 0 \end{array} \Leftrightarrow \begin{array}{c} \min(f)(\Delta) \leq 0 \leq \max(f)(\Delta) \\ \min(g)(\Delta) \leq 0 \leq \max(g)(\Delta) \end{array}$$

only $\Rightarrow$-part is valid.

Since $fis(0,0,r)$ is not bound we can apply (2.23). So, from the given system $f = 0, g = 0$ we pass to the new one:

$$f = 0, g = 0, \quad f'_x g'_y - f'_y g'_x = 0$$

i.e. to

$$y - x^2 = 0, \quad y = 0, x = 0$$

Now the members of the new $fis(0,0,r)$ are in turn

1, 1,...

i.e. generally $fis(0,0,r) = 1$.

[11] Each time a cooresponding condition of the type (2.22) 2^0 should be fulfiled.
[12] The pair (0,0) is the unique solution in D. Also the bisection way is used

Example 2.6 *Let $f(x) = 0$ be the equation*

$$x^3 - 3x^2 + 3x - 1 = 0$$

and $D = [1, 2]$.

Obviously, 1 is a triple solution of this equation. Using diadic tree denote by

$$\Delta = [\alpha, \alpha + \frac{1}{2^r}], \text{ with } \alpha = 1 + \frac{i}{2^r}, i = 0, 1, ..., 2^r - 1$$

any cell in the r-th step. Then for $m(f)$, $M(f)$ we have the following formulas

$$m(f)(\Delta) = \alpha^3 - 3\left(\alpha + \frac{1}{2^r}\right)^2 + 3\alpha - 1,$$

$$M(f)(\Delta) = \left(\alpha + \frac{1}{2^r}\right)^3 - 3\alpha^2 + 3\left(\alpha + \frac{1}{2^r}\right) - 1$$

The condition $M(f)(\Delta) \geq 0$ is always fulfilled for

$$M(f)(\Delta) \geq \alpha^3 - 3\alpha^2 + 3\alpha - 1 \geq (\alpha - 1)^3 \geq 0$$

The condition $m(f)(\Delta) \leq 0$ can be rewritten in this way

(*1) $$(\alpha - 1)^3 \leq \frac{3}{2^r}\left(2\alpha + \frac{1}{2^r}\right)$$

This inequality holds provided that

$$(\alpha - 1)^3 \leq 6 \cdot \frac{1}{2^r}$$

Putting $\alpha = 1 + \frac{1}{2^r}$ we get the inequality

(*2) $$i^3 \leq 6 \cdot 2^{2r} \quad (i = 0, 1, ..., 2^r - 1)$$

Denote by $F_1(r)$ the number of all i-solution of that inequality. This $F_1(r)$ is not bounded, the same is true for $fis(1, r)$ since $fis(c, r) \geq F_1(r)$. So, by (2.23) from the equation $f(x) = 0$ we pass to the system

$$f(x) = 0, f'(x) = 0$$

i.e. to the system

$$x^3 - 3x^2 + 3x - 1 = 0, x^2 - 2x + 1 = 0$$

Now we have a new m-M pair

$$m(f')(\Delta) = \alpha^2 - 2\left(\alpha + \frac{1}{2^r}\right) + 1, M(f')(\Delta) = \left(\alpha + \frac{1}{2^r}\right)^2 - 2\alpha + 1$$

Since $M(f')(\Delta) \geq \alpha^2 - 2\alpha + 1 \geq 0$ the condition $M(f')(\Delta) \geq 0$ holds. Consequently, a cell Δ is feasible iff

(*3) $$m(f)(\Delta) \leq, m(f')(\Delta) \leq 0$$

The condition $m(f')(\Delta) \leq 0$ can be rewriten in this way

$$(\alpha - 1)^2 \leq 2 \cdot \frac{1}{2^r}$$

i.e.

(*4) $$i^2 \leq 2^{r+1} \quad (i = 0, 1, ..., 2^r - 1)$$

Notice that (*4) implies (*2), and since (*2) implies (*1) we conclude that (*4) implies (*1). In other words feasibility condition (*3) can be reduced to (*4). Denote by[13] $F_2(r)$ the number of all i-solution of (*4). First we point out that $F_2(r) < F_1(r)$, which means that the transition from $f(x) = 0$ to the system $f(x) = 0, f'(x) = 0$ profoundly[14] diminished $fis(1, r)$. However, since $F_2(r)$ is not bounded we again use (2.23). So, from the system $f(x) = 0, f'(x) = 0$ we pass to the system $f(x) = 0, f'(x) = 0, f''(x) = 0$, i.e. to the system

$$x^3 - 3x^2 + 3x - 1 = 0, \; x^2 - 2x + 1 = 0, \; x - 1 = 0$$

But now $fis(1, r)$ will be 1, since only the cell $\left[1 + \frac{1}{2^r}\right]$ is feasible.

About the Example 2.6 we also notice the following. For the given equation $f(x) = 0$ an m-M pair $m(f)$, $M(f)$ can be defined by these equalities

$$m(f)(\Delta) = \min(f)(\Delta) = \alpha^3 - 3\alpha^2 + 3\alpha - 1$$
$$M(f)(\Delta) = \max(f)(\Delta) = \beta^3 - 3\beta^2 + 3\beta - 1$$
$$(\text{where } \Delta = [\alpha, \beta])$$

Then in each step r the number $fis(1, r)$ will be 1.

In the following example we consider a system (2.7) with $k < n$.

Example 2.7 *Equation in $(x, y) \in [-4, 4] \times [-3, 3]$*

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

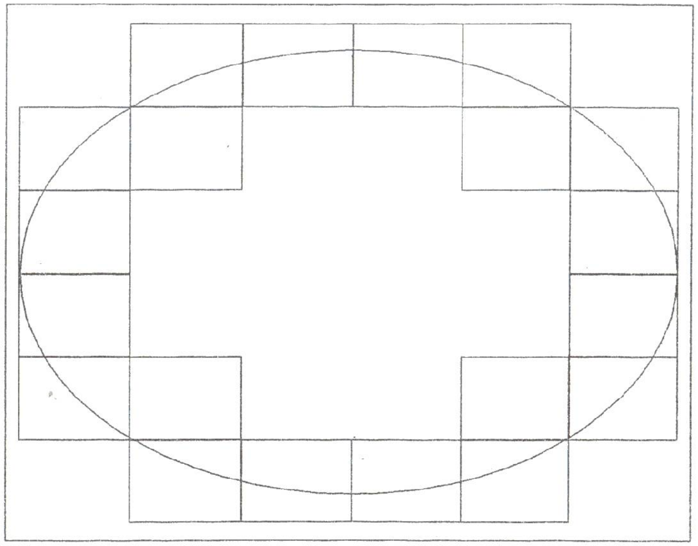
This equation has infinity number of solutions. Using diadic tree the numbers of feasible cells are in turn

$$4, \; 12, \; 28, \; 56, \; 112, \; 224, \; 448, ...$$

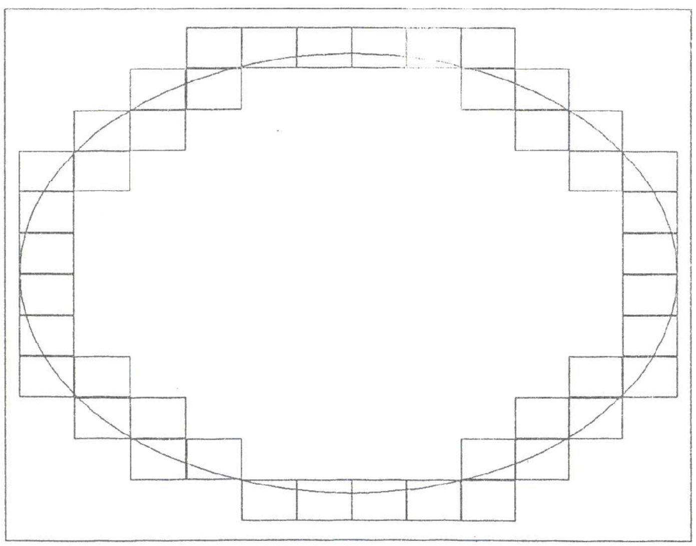
and corresponding drawings are

[13] Thus $F_2(r)$ is the new $fis(1, r)$.
[14] Namely, $F_1(r)/F_2(r) \to 0$ when $r \to \infty$.

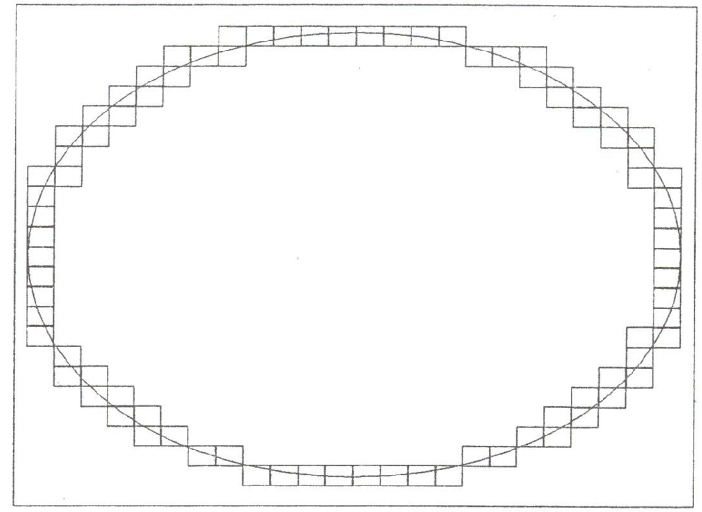
Step 3:



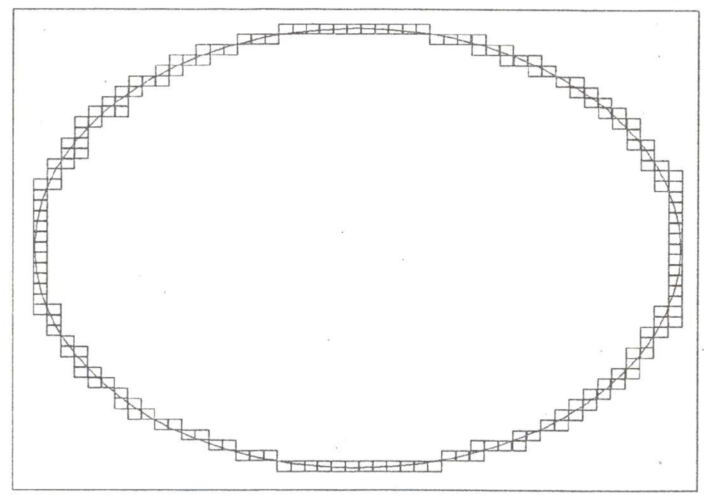
Step 4:



Step 5:



Step 6:



At the end we give an example in which also solutional cells appear.

Example 2.8 *Inequality*

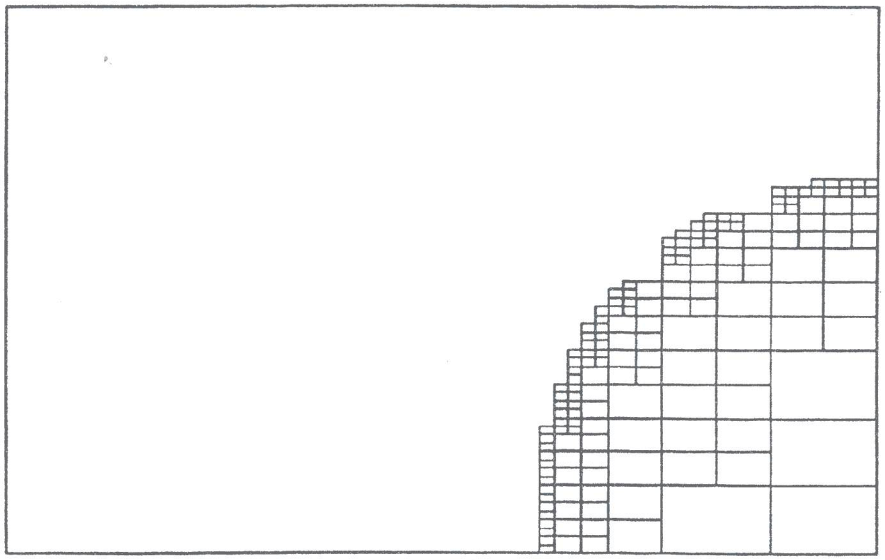
$$2 + e \cdot (x + y + 2) - e^{x+1} - e^{y+1} \geq 0$$

in $(x, y) \in [0, 2.4] \times [0, 1.4]$.

Using diadic tree the numbers of indetermined and solutional cells are in turn

indetermined: 4, 15, 31, 67, 139, 275, ...
solutional: 0, 0, 13, 38, 102, 239, ...

Here is the corresponding drawing in the step 6



3. n-DIMENSIONAL INTEGRALS, INFINITE SUMS; THEIR m-M PAIRS

For functions in whose definitions the notions of n-dimensional integrals or infinite sums are involved the following questions are studied: how to calculate their values and how to determine m-M pairs.

1. Let $D = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$ be a given n-dimensional segment and let S be the set of all solutions in $x_1, ..., x_n$ of the following system of inequalities

(3.1) $$f_1(x_1, ..., x_n) \geq 0, ..., f_k(x_1, ..., x_n) \geq 0 \quad ((x_1, ..., x_n) \in D)$$

where $f_1, ..., f_k : D \to \mathbb{R}$ are given m-M functions. Denote by $S_k(S)$ the subset of the set S determined by the following additional condition

(3.2) $$f_1(x_1, ..., x_n) = 0 \vee f_2(x_1, ..., x_n) = 0 \vee ... \vee f_k(x_1, ..., x_n) = 0$$

Assuming that the Jordan measure of the set $S_k(S)$ is zero we look for the value of the Riemann integral

(3.3) $$I = \int \cdots \int_S g(x_1, ..., x_n) dx_1 ... dx_n$$

where $g : D \to \mathbb{R}$ is a given function having some Lipschitz's m-M pair. The procedure we are going to state is a continuation of the procedure (2.6) concerning system (3.1). In connection with the sets S_r, U_r (see (2.6)) denote by S'_r, U'_r the sets of all r-products P_r whose unions are the sets S_r, U_r respectively. We shall also use the following well-known functions[1)]

(3.4) $$mi(x) = \min\{0, x\}, ma(x) = \max\{0, x\}$$

Theorem 3.1. *Let $S \neq \emptyset$ and let $\langle L_r(I)\rangle, \langle R_r(I)\rangle$ be two r-sequences defined by the following equalities*

(3.5)
$$L_r(I) = \sum_{P \in S'_r} m(g)(P) \cdot V(P) + \sum_{P \in U'_r} mi\,(m(g)(P)) \cdot V(P)$$
$$R_r(I) = \sum_{P \in S'_r} M(g)(P) \cdot V(P) + \sum_{P \in S'_r} ma\,(M(g)(P)) \cdot V(P)$$

($r \in \mathbb{N}$, $V(P)$ is the volume of P)

where $\langle m(g), M(g)\rangle$ is some Lipschitz's m-M pair. Then for every $r \in \mathbb{N}$ the following double inequality

(3.6) $$L_r(I) \leq I \leq R_r(I)$$

is true. Moreover, the equality

(3.7) $$\lim_{r \to \infty} (R_r(I) - L_r(I)) = 0$$

1) In mathematical literature usually the denotations x^-, x^+ are used respectively

holds

Proof. Based on the solving procedure (2.6) the following double inclusion

$$S_r \subseteq S \subseteq S_r \cup U_r \quad (r = 0, 1, ...)$$

holds. Consider the sums

$$I'_r = \sum_{P \in S'_r} g(\xi_1, ..., \xi_n) \cdot V(P), \quad I''_r = \sum_{P \in S'_r \cup U'_r} g(\xi_1, ..., \xi_n) \cdot V(P)$$

where $(\xi_1, ..., \xi_n)$ may be any element of the product P. First we prove the following inequalities

$$L_r(I) \le I'_r \le R_r(I), \quad L_r(I) \le I''_r \le R_r(I)$$

Indeed, one proof of the inequality $L_r(I) \le I'_r$ reads:

$$\begin{aligned} I'_r &= \sum_{P \in S'_r} g(\xi_1, ..., \xi_n) \cdot V(P) \\ &\ge \sum_{P \in S'_r} m(g)(P) \cdot V(P) \qquad (\text{for } g(\xi_1, ..., \xi_n) \ge m(g)(P)) \\ &\ge \sum_{P \in S'_r} m(g)(P) \cdot V(P) + \sum_{P \in U'_r} mi\,(m(g)(P)) \cdot V(P) \quad (\text{for } mi(x) \le 0) \\ &= L_r(I) \end{aligned}$$

In a similar way one can prove the remaining inequalities. To complete the proof we shall prove (3.7). Consider the equality

$$\begin{aligned} R_r(I) - L_r(I) = &\sum_{P \in S'_r} (M(g)(P) - m(g)(P)) \cdot V(P) \\ &+ \sum_{P \in U'_r} (ma\,(M(g)(P)) - mi\,(m(g)(P)))) \cdot V(P) \end{aligned}$$

Using Definition 1.8 it is easy to prove that

$$\lim_{r \to \infty} \sum_{P \in S'_r} (M(g)(P) - m(g)(P)) \cdot V(P) = 0$$

Next bearing in mind (1.29) to complete the proof it suffices to prove the following equality

$$(*) \qquad \lim_{r \to \infty} \sum_{P \in U'_r} V(P) = 0$$

The set $S_k(S)$ is defined by conditions (3.1), (3.2) which can briefly rewritten as follows

$$(3.8) \qquad (\forall i)\, f_i(x_1, ..., x_n) \ge 0, \quad (\exists i)\, f_i(x_1, ..., x_n) = 0, \quad (x_1, ..., x_n) \in D$$

The set $S_k(S)$ can be determined by a procedre similar to (2.6), more precisely said, by the procedure (5.1) (see also Theorem 4.4, part (jj)). Then according to Definition 4.2 a product $P_r \in \mathcal{D}_r(D)$ is feasible in the sense of (3.8) if and only if the following conditions are satisfied

$$(\forall i)\, M(f_i)(P_r) \ge 0, \quad (\exists i)\, (m(f_i)(P_r) \le 0 \le M(f_i)(P_r))$$

which is logically equivalent to the conditions

$$(\forall i)\, M(f_i)(P_r) \le 0, \quad (\exists i)\, m(f_i)(P_r) \le 0$$

Using Theorem 4.4, part (jj), i.e. the equality

$$S_k(S) = \bigcap_{r \in K} F_r\,((3.8))$$

and the assumption that the Jordan measure of $S_k(S)$ is zero we conclude that

$$\lim_{r \to \infty} \sum_{P_r \subseteq F_r((3.8))} V(P) = 0$$

Finally since $U_r \subseteq F_r\,((3.8))$ this equality implies equality (*). The proof is completed.

Notice that one disadvantage of the approximative formula (3.5) is that its two sides should eventually be calculated for all values of r, and each of these calculations is independent. However, this difficulty can be mitigated in the case when the function g is suitable for effective integration respective to n-dimensional intervales. Namely, in that case in formula (3.5) the parts

$$\sum_{P \in S'_r} m(g)(P) \cdot V(P), \quad \sum_{P \in S'_r} M(g)(P) \cdot V(P)$$

can be replaced by

$$\sum_{P \in S'_r} \int \cdots \int_P g(x_1, ..., x_n) dx_1 ... dx_n$$

In accordance with this in the eventual further calculation, having in mind procedure (2.6), we retain the members $P \in S'_r$ and gradualy diminish only members of the set U'_r in order to separate new solutional cells; then on them we again calculate the integrals of the function g, etc.

Remark 3.1. *It is supposed that $\mathcal{D}[D]$ is a call-tree. Concerning the sequences $\langle L_r(I) \rangle$, $\langle R_r(I) \rangle$ we mention the following. In general, they are not monotone sequences. However, using the idea of Definition 1.7 and supposing that $\mathcal{D}(D)$ is a tree we can make two new $\langle \overline{L_r}(I) \rangle$, $\langle \overline{R_r}(I) \rangle$, as follows*

$$\begin{aligned} \overline{L_0}(I) &= L_0(I), \quad \overline{L_{r+1}}(I) = \max\left(\overline{L_r}(I), L_r(I)\right) \\ \overline{R_0}(I) &= R_0(I), \quad \overline{R_{r+1}}(I) = \min\left(\overline{R_r}(I), R_r(I)\right) \quad (r \in \mathbb{N}) \end{aligned}$$

These sequences are monotone and also satisfy conditions of the type (3.6) and (3.7). Additionally, notice that the sequences $\langle L_r(I)\rangle$, $\langle R_r(I)\rangle$ are monotone under the following condition

(**) *The m-M pair $\langle m(g), M(g)\rangle$ is monotone.*

The proof is more or less straightforward.

Example 3.1. *For the integral I:*

$$\iint_{Cond(x,y)} xydxdy$$

where $Cond(x,y)$ reads

$$0 \le x \le 2,\ 0 \le y \le 2, \qquad 2 + e\cdot(x+y+2) \ge e^{x+1} + e^{y+1}$$

by using 4-tree[2] *we have the following results*

Step 1:	$0.0000000000000000 \le I \le 0.5625000000000000$
Step 2:	$0.0278320312500000 \le I \le 0.2424316406250000$
Step 3:	$0.0950307846069336 \le I \le 0.1581497192382812$
Step 4:	$0.1179245151579380 \le I \le 0.1341890022158623$
Step 5:	$0.1239582093403442 \le I \le 0.1281216432544170$
Step 6:	$0.1255188694198068 \le I \le 0.1265613023800256$
Step 7:	$0.1259090350460332 \le I \le 0.1261702301487544$
Step 8:	$0.1259090350460332 \le I \le 0.1259090350460332$
Step 9:	$0.1259090350460332 \le I \le 0.1259090350460332$

2. Now we are going to extend the notion of m-M pairs to the case of the functions in whose definitions some Riemann integrals or infinite sums are involved.

One such example is the function $f : [a,b] \to \mathbb{R}$ defined by the equality

$$f(x) = \int_a^x g(t)dt \tag{3.9}$$

where $g : [a,b] \to \mathbb{R}$ is a function having a Lipschitz's m-M pair.

Let $\Delta = [\alpha,\beta]$ be any subsegment of the segment $[a,b]$. Then an m-M pair $\langle m(f)(\Delta), M(f)(\Delta)\rangle$ can be defined as follows

$$\begin{aligned} m(f)(\Delta) &= \left(\sum_{k=1}^{l(\Delta)} m(g)(\Delta_k)\,diam\ \Delta_k\right) - (M(|g|)(\Delta))diam\ \Delta \\ M(f)(\Delta) &= \left(\sum_{k=1}^{l(\Delta)} M(g)(\Delta_k)\,diam\ \Delta_k\right) + (M(|g|)(\Delta))diam\ \Delta \end{aligned} \tag{3.10}$$

[2] In every step one dimensional cells are divided in 4 equal subcells

providing that the segments $\Delta_1, ..., \Delta_{l(\Delta)}$ satisfy the following conditions:

1^0 $\Delta_1 \cup ... \cup \Delta_{l(\Delta)} = [\alpha,\beta]$,

2^0 The interiors of any two segments Δ_i, Δ_j, with $i \ne j$ are mutually disjoint.

3^0 $\lim_{diam\ \Delta_k} \max diam\ \Delta_k = 0$

Proof. Let x be any element of Δ. Then we have

$$\begin{aligned} f(x) &= \int_a^\alpha g(t)dt + \int_\alpha^x g(t)dt \\ &\le \sum_{k=1}^{l(\Delta)} M(g)(\Delta_k)diam\ \Delta_k + \int_\alpha^\beta |g(t)|\,dt \\ &\le \sum_{k=1}^{l(\Delta)} M(g)(\Delta_k)diam\ \Delta_k + M(|g|)(\Delta)diam\ \Delta \end{aligned}$$

So, $M(f)(\Delta)$ defined by (3.10) really is an upper bound of f when $x \in \Delta$. Similarly one can prove that $m(f)(\Delta)$ is a lower bound of f when $x \in \Delta$.

In virtue of (1.29) there exist $\varepsilon > 0$, $K > 0$ such that $M(|g|)(\Delta) < K$ whenever $diam\ \Delta < \varepsilon$. Accordingly in view of the equality

$$M(f)(\Delta) - m(f)(\Delta) = \sum_{k=1}^{l(\Delta)} (M(g)(\Delta_k) - m(g)(\Delta_k))\,diam\Delta_k + 2M(|g|)(\Delta)diam\Delta$$

to complete the proof it suffices to prove that

$$lim_{diam\ \Delta\to 0} \sum_{k=1}^{l(\Delta)} (M(g)(\Delta_k) - m(g)(\Delta_k))\,diam\ \Delta_k = 0$$

This follows directly from the assumption that $\langle m(g), M(g)\rangle$ is a Lipshitz's m-M pair. The proof is complete.

Now consider the function $f : [a,b] \to \mathbb{R}$ defined by the following equality

$$f(x) = \sum_{i=1}^{\infty} g_i(x) \tag{3.11}$$

where $g_i(x) : [a,b] \to \mathbb{R}$ are given m-M functions. Then it can be easily proved that one m-M pair of f can be determined by the following formulas

$$m(f)(\Delta) = \sum_{i=1}^{\infty} m(g_i)(\Delta), \quad M(f)(\Delta) = \sum_{i=1}^{\infty} M(g_i)(\Delta), \quad (\Delta \subseteq [a,b]) \tag{3.12}$$

providing that both infinite sums are Δ-uniformly convergent. In connection with this let us additionally suppose that for all $x \in [a, b]$ an inequality of the form

$$\left|\sum_{i=n}^{\infty} g_i(x)\right| \leq h_n$$

is satisfied where $\langle h_n \rangle$ is some sequence with the property $h_n \to 0$ if $n \to \infty$. Then one m-M pair for the function f can be defined as follows

$$m(f)(\Delta) = \sum_{i=1}^{l(diam\ \Delta)} m(g_i)(\Delta) - h_{n(diam\ \Delta)}$$

(3.13)

$$M(f)(\Delta) = \sum_{i=1}^{l(diam\ \Delta)} M(g_i)(\Delta) + h_{n(diam\ \Delta)}$$

where $l(diam\ \Delta) \in \mathbb{N}$ is a function of $diam\ \Delta$ with the property

$$\lim_{diam\ \Delta \to 0} l(diam\ \Delta) = \infty.$$

For instance, a function l can be determined by: $l(t) = \left[\frac{1}{t}\right]$, where $t > 0$.

Example 3.2. *Let f be a function defined by the equality*

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{x + 2^i}$$

Concerning the equation $f(x) = c$, with $a \leq x \leq b$ where a, b, c are constants we have the following results (diadic tree is used)

C a s e $c = 1.5, a = 0, b = 1$

In the step 20 we obtain the following double inequality

$$0.54416 \leq x \leq 0.54417$$

The numbers $fis(r)$ $(r = 1, 2, ..., 20)$ are in turn

1, 2, 3, 3, 2, 3, 2, 2, 3, 2, 3, 3, 2, 2, 2, 3, 2, 3, 2, 3

C a s e $c = 1.5, a = 0.6, b = 1$

Step 1 $fis(1) = 1$; Step 2 $fis(2) = 1$; Step 3 $fis(3) = 0$
Conclusion: $f(x) = c$ has no solutions

Similarly, concerning the equation $f(x) = x$, with $a \leq x \leq b$ we have the following results

C a s e $a = 0, b = 1$. In the third step $fis(r) = 0$, so the given equation has no solutions.

C a s e $a = 0, b = 2$. The $fis(r)$ $(r = 1, 2, ..., 20)$ is 1 or 2. In the step 20 we obtain the following double inequality

$$1.19055 \leq x \leq 1.19056$$

Example 3.3. *Let f be a function defined by* $f(x) = \int_0^x \frac{e^t}{1+t} dt$

Concerning the equation $f(x) = C$, with $A \leq x \leq B$ where A, B, C are constants we have the following results (diadic tree is used)

Case : C=1, [A,B]=[0,1]

Step 1, fis=1
$0 \leq x \leq 1$
Step 2, fis=1
$0.5 \leq x \leq 1$
Step 3, fis=2
$0.5 \leq x \leq 1$
Step 4, fis=2
$0.625 \leq x \leq 1$
Step 5, fis=4
$0.75 \leq x \leq 1$
Step 6, fis=3
$0.8435 \leq x \leq 0.9375$
Step 7, fis=3
$0.875 \leq x \leq 0.921875$
Step 8, fis=3
$0.89065 \leq x \leq 0.9140625$
Step 9, fis=1
$0.8984375 \leq x \leq 0.91015625$
Step 10, fis=3
$0.90234375 \leq x \leq 0.908203125$
Step 11, fis=3
$0.903320313 \leq x \leq 0.90625$
Step 12, fis=4
$0.904296875 \leq x \leq 0.90625$
Step 13, fis=4
$0.905029297 \leq x \leq 0.905761719$

Case : C=1, [A,B]=[1,20]

Step 1, fis=2
$1 \leq x \leq 20$
Step 2, fis=2
$1 \leq x \leq 10.5$
Step 3, fis=2
$1 \leq x \leq 5.75$
Step 4, fis=2
$1 \leq x \leq 3.375$
Step 5, fis=2
$1 \leq x \leq 2.1875$
Step 6, fis=1
$1 \leq x \leq 1.296875$
Step 7, fis=1
$1 \leq x \leq 1.1484375$
Step 8, fis=0
no solution

Case: $C = 1, [A, B] = [3, 100]$

Step 1, fis=1
$3 \leq x \leq 30$
Step 2, fis=2
$3 \leq x \leq 16.5$
Step 3, fis=2
$3 \leq x \leq 9.75$
Step 4, fis=1
$3 \leq x \leq 4.6875$
Step 5, fis=0
no solution

Similarly, concerning the equation $f(x) = 1/x$ with $A \le x \le B$ (A, B are constants) we have the following results

Case : C=1, [A,B]=[0.2,1]

Step 1, fis=1
$0.2 \le x \le 1$

Step 2, fis=1
$0.6 \le x \le 1$

Step 3, fis=1
$0.6 \le x \le 1$

Step 4, fis=2
$0.8 \le x \le 1$

Step 5, fis=2
$0.9 \le x \le 1$

Step 6, fis=2
$0.925 \le x \le 0.975$

Step 7, fis=2
$0.937 \le x \le 0.9625$

Step 8, fis=1
$0.94375 \le x \le 0.95$

Step 9, fis=2
$0.94375 \le x \le 0.95$

Step 10, fis=2
$0.946875 \le x \le 0.95$

Step 11, fis=2
$0.946875 \le x \le 0.9484375$

Step 12, fis=2
$0.947265625 \le x \le 0.948046875$

Case : C=1, [A,B]=[1,20]

Step 1, fis=1
$1 \le x \le 20$

Step 2, fis=1
$1 \le x \le 10.5$

Step 3, fis=1
$1 \le x \le 5.75$

Step 4, fis=1
$1 \le x \le 3.375$

Step 5, fis=1
$1 \le x \le 2.1877$

Step 6, fis=1
$1 \le x \le 1.59375$

Step 7, fis=1
$1 \le x \le 1.296875$

Step 8, fis=1
$1 \le x \le 1.484375$

Step 9, fis=0
no solution

4. m-M PAIRS OF THE FIRST ORDER FORMULAS; SET-THEORETICAL INTERPRETATION

In this section we study a way in which various informations on real numbers, expressed by the first order-logic formulas, can be translated into the corresponding information involving the notion of m-M pair. More precisely formulas (4.9), (4.10) are proved.

1. First let a class of certain m-M functions of the type

$$f : D \to \mathbb{R} \qquad (D = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n)$$

be given, where number n (>0) and the domain D for each function may be arbitrarily chosen. In connection with such functions we consider a class of the first order $<, \le$-formulas. The definition given bellow slightly differs from usual definitions of the first order formulas (see for instance [6]).

First, an atomic $<, \le$-formula is any formula of the form

$$\text{(4.1)} \qquad f(y_1, ..., y_p)\ \rho\ g(z_1, ..., z_q) \qquad (\rho \text{ may be} < \text{or} \le)$$

where

$$f : [a_1, b_1] \times ... \times [a_p, b_p] \to \mathbb{R}, \qquad g : [c_1, d_1] \times ... \times [c_q, d_q] \to \mathbb{R}$$

are some m-M functions[1)] and elements of the given class, while $y_1, ..., y_p, z_1, ..., z_q$ are variables which do not need to be mutually different[2)]. Further, generally a $<, \le$-formula is any formula built up from some atomic $<, \le$-formula using in a finite number of steps the logical connectives $\wedge, \vee, \neg$ and the quantifiers of the form $(\forall v \in I(v))$, $(\exists v \in I(v))$, where v is a variable and $I(v)$ is the set called <u>segment</u> of v, the notion to be explained bellow. In order to avoid some technical difficulties about the way of building the $<, \le$-formulas we generally assume the following[3)]

(4.2) *In any $<, \le$-formula all bounded variables are mutually different and additionally none variable can be both free and bounded. Further in the case of formulas of the form*

$$(q\ v \in I(v))\ \Psi \qquad (q \text{ is } \forall \text{ or } \exists; \quad v \text{ may be any variable})$$

the variable v must be a free variable of the subformula Ψ.

Now we shall explain the notion of 'segment of v'.

[1)] More precisely said, (4.1) is a string which f, g are functional symbols, ρ is a relational symbol and $y_1, ..., y_p, z_1, ..., z_q$ are some variables.

[2)] Examples of such formulas are:

$$f(x, x, y) < g(y, z, x), \qquad h(x, z) \le g(x, x, y)$$

[3)] As a matter of fact $<, \le$-formula built up in a standard way is logically equivalent to some $<, \le$-formula with property (4.2).

In the case of atomic formulas of the type (4.1) if all variables y_i, z_j are mutually different we take these definitions

$$I(y_i) = [a_i, b_i], \quad I(z_j) = [c_j, d_j]$$

However, if for example the variables

$$y_{i_1}, ..., y_{i_r}, z_{j_1}, ..., z_{j_s}$$

are mutually equal but different from all other variables then their segment is defined as the following intersection

$$[a_{i_1}, b_{i_1}] \cap ... \cap [a_{i_r}, b_{i_r}] \cap [c_{j_1}, d_{j_1}] \cap ... \cap [c_{j_s}, d_{j_s}]$$

Let further φ be any $<, \leq$-formula and let v be its free variable. Denote by $\Psi_1, ..., \Psi_t$ all atomic subformulas of φ containing the variable v. Then $I(v)$ is the intersection of all segments of v which are already related to the subformulas $\Psi_1, ..., \Psi_t$. Finally, if v is a bounded variable of φ then according to (4.2) there is exactly one subformula of φ which has the form

$$(q\ v \in I(v))\ \Psi(v) \quad (q \text{ is } \forall \text{ or } \exists)$$

and v is a free variable of $\Psi(v)$. Then, by definition, for $I(v)$ we take the segment of v with respect to the formula $\Psi(v)$.

Example 4.1. *Let*

$$f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}, \quad g : [c_1, d_1] \times [c_2, d_2] \to \mathbb{R}, \quad h : [e_1, f_1] \times [e_2, f_2] \to \mathbb{R}$$

be some given m-M functions. Then

$$(4.3) \qquad (\forall x \in I(x))\ (\exists y \in I(y))\ (f(x,z) < g(z,y) \ \wedge \ h(x,y) < f(z,z))$$

is an example of $<, \leq$-formula for which we have the equalities

$$I(x) = [a_1, b_1] \cap [e_1, f_1], \quad I(y) = [c_2, d_2] \cap [e_2, f_2], \quad I(z) = [a_1, b_1] \cap [a_2, b_2] \cap [c_1, d_1]$$

2. Let now φ be a given $<, \leq$-formula whose all variables are among the variables $x_1, ..., x_m$. Suppose that to each segment $I(x_i)$ $(i = 1, ..., m)$ one cell-decomposition $\mathcal{D}(I(x_i))$ is assigned (see Definition 1.2). We are going to define the notion of m-M pair of the formula φ with respect to the cell-decompositions $\mathcal{D}(I(x_i))$, $(i = 1, ..., n)$. The m-M pair will be defined as a particular sequence of ordered pairs

$$\langle m_0(\varphi), M_0(\varphi) \rangle, ..., \langle m_r(\varphi), M_r(\varphi) \rangle, ...$$

Their definition, i.e. Definition 4.1 bellow, has sintactical nature. Starting with a given φ we apply this definition recursively. We point out that during this process each bounded variable say x_i will be replaced by a new variable X_i.

Definition 4.1.

(i) $m_r(f(y_1, ..., y_p)\rho g(z_1, ..., z_q)) = m(f)(y_1' \times ... \times y_p')\rho M(g)(z_1' \times ... \times z_q')$
$M_r(f(y_1, ..., y_p)\rho g(z_1, ..., z_q)) = M(f)(y_1' \times ... \times y_p')\rho m(g)(z_1' \times ... \times z_q')$
(ρ *may be* $<$ *or* $\leq$).

Variables $y_1, ..., y_p, z_1, ..., z_q$ *are some of* $x_1, ..., x_m, X_1, ..., X_m$. *Denotation* $'$ *means the following*

$$x_i' \text{ denotes } C_r(x_i), \text{ while } X_i' \text{ denotes } X_i$$

(ii) $m_r(\alpha \wedge \beta) = m_r(\alpha) \wedge m_r(\beta)$; $M_r(\alpha \wedge \beta) = M_r(\alpha) \wedge M_r(\beta)$

(iii) $m_r(\alpha \vee \beta) = m_r(\alpha) \vee m_r(\beta)$, $M_r(\alpha \vee \beta) = M_r(\alpha) \vee M_r(\beta)$

(iv) $m(\neg\alpha) = \neg M_r(\alpha)$, $M(\neg\alpha) = \neg m_r(\alpha)$

(v) *Let $\alpha(x_i)$ be a formula having x_i as a free variable and q be a quantifier $\forall$ or $\exists$. Then we have the following equalities*[1]

$$m_r((q\ x_i \in I(x_i))\ \alpha(x_i)) = (q\ X_i \in \mathcal{D}_r(I(x_i)))\ m_r(\alpha(X_i))$$

$$M_r((q\ x_i \in I(x_i))\ \alpha(x_i)) = (q\ X_i \in \mathcal{D}_r(I(x_i)))\ M_r(\alpha(X_i))$$

For instance, if φ is the formula $(\forall x_2)\ f(x_1, x_2) < g(x_2, x_3)$ then $m_r(\varphi)$ $(r = 0, 1, ...)$ by Definition 4.1 can be constructed as follows

$m_r((\forall x_2)\ f(x_1, x_2) < g(x_2, x_3))$
$= (\forall X_2 \in \mathcal{D}_r(I(x_2))\ m_r(f(x_1, X_2) < g(X_2, x_3))$
Part (v) of the definition
$= (\forall X_2 \in \mathcal{D}_r(I(x_2))\ m(f)(C_r(x_1) \times X_2) < M(g)(X_2 \times C_r(x_3))$
Part (i) of the definition

In a similar way one can obtain the following formula for $M_r(\varphi)$:

$M_r((\forall x_2)\ f(x_1, x_2) < g(x_2, x_3))$
$= (\forall X_2 \in \mathcal{D}_r(I(x_2)))\ M(f)(C_r(x_1) \times X_2) < m(g)(X_2 \times C_r(x_3))$

Notice that in an obvious way these $m_r(\varphi)$, $M_r(\varphi)$ can also be treated as some first-order formulas. Then X_2 is a bounded variable for both of them and the symbols $C_r(x_1), C_r(x_3)$ should be taken as their free variables. Accordingly, if we denote the formula φ by $\varphi(x_1, x_3)$, emphasizing that x_1 and x_3 are free variable of φ, then it is natural that $m_r(\varphi)$, $M_r(\varphi)$ are denoted by

$$m_r(\varphi)\ (C_r(x_1), C_r(x_3))\ , M_r(\varphi)\ (C_r(x_1), C_r(x_3))$$

respectively. Generally we are going to apply similar denotations.

Theorem 4.1. *Let $\varphi(x_1, .., x_m)$ (with $m \geq 0$) be a $<, \leq$-formula whose all free variables are among $x_1, ..., x_m$. Then for every $r \in \mathbb{N}$ the following double implication is true*

$$(4.4) \quad M_r(\varphi)\ (C_r(x_1), ..., C_r(x_m)) \Rightarrow \varphi(x_1, ..., x_m) \Rightarrow m_r(\varphi)\ (C_r(x_1), ..., C_r(x_m))$$

[1] Notice that in this step the variable x_i is replaced by X_i.

provided that the variabled $x_1, ..., x_m$ have any values from their segments $I(x_1), ..., I(x_m)$ respectively.

Proof. Denote by $l(\varphi)$ the number of all logical symbols $\wedge, \vee, \neg, \forall, \exists$ occuring in the formula[1] $\varphi(x_1, ..., x_m)$. The proof is by induction on $l(\varphi)$.

If $l(\varphi) = 0$ then the formula φ has the form

$$f(y_1, ..., y_p)\ \rho\ g(z_1, ..., z_q) \quad (\rho \text{ is } < \text{ or } \leq)$$

with $\{y_1, ..., y_p, z_1, ..., z_q\} = \{x_1, ..., x_m\}$ and (4.4) reduces to the following double implication

$$M(f)\,(C_r(y_1) \times ... \times C_r(y_p))\ \ \rho\ m(g)\,(C_r(z_1) \times ... \times C_r(z_q))$$
$$\Rightarrow f(y_1, ..., y_p)\ \rho\ g(z_1, ..., z_q) \Rightarrow$$
$$m(f)\,(C_r(y_1) \times ... \times C_r(y_p))\ \ \rho\ M(g)\,(C_r(z_1) \times ... \times C_r(z_q))$$

which follows directly from axiom (0.1).

Let $l(\varphi) > 0$. The formula $\varphi(x_1, ..., x_m)$ can have one of the forms

$$1^0 \alpha \wedge \beta, \quad 2^0 \alpha \vee \beta, \quad 3^0 \neg\alpha$$
$$4^0\ (\exists x \in I(x))\, \alpha(x, x_1, ..., x_m), \quad 5^0\ (\forall x \in I(x))\, \alpha(x, x_1, ..., x_m)$$

In the case 1^0 we have

$M_r(\alpha \wedge \beta)\,(C_r(x_1), ..., C_r(x_m))$
$\Rightarrow$ $M_r(\alpha)\,(C_r(x_1), ..., C_r(x_m)) \wedge M_r(\beta)\,(C_r(x_1), ..., C_r(x_m))$
(By Definition 4.1)
$\Rightarrow$ $\alpha(x_1, ..., x_m) \wedge \beta(x_1, ..., x_m)$
(By induction hypothesis)

So in the case 1^0 the implication $M_r(\varphi) \Rightarrow \varphi$ is proved. Similarly the implication $\varphi \Rightarrow m_r(\varphi)$ can be proved.

Proof in the case 2^0 is similar. In the case 3^0 for the implication $M_r(\varphi) \Rightarrow \varphi$ we have

$M_r(\neg\alpha)\,(C_r(x_1), ..., C_r(x_m))$
$\Rightarrow$ $\neg m_r(\alpha)\,(C_r(x_1), ..., C_r(x_m))$ (By Definition 4.1)
$\Rightarrow$ $\neg\alpha(x_1, ..., x_m)$ (By induction hypothesis, i.e. by the implication $\alpha(x_1, ..., x_m) \Rightarrow m(\alpha)\,(C_r(x_1), ..., C_r(x_m))$)

Similarly the implication $\varphi \Rightarrow m_r(\varphi)$ can be proved. In the case 4^0 for the implication $M_r(\varphi) \Rightarrow \varphi$ we have

$(\exists X \in \mathcal{D}_r\,(I(x)))\ M_r(\alpha)\,(X, C_r(x_1), ..., C_r(x_m))$
$\Rightarrow$ $M_r(\alpha)\,(X_0, C_r(x_1), ..., C_r(x_m))$
(X_0 is some fixed element of $\mathcal{D}_r\,(I(x))$).

[1]It is supposed that during the proof formula φ none of the brackets is omitted. So, for instance, instead of $A \wedge B \wedge C$ we have $((A \wedge B) \wedge C)$.

$\Rightarrow$ $M_r(\alpha)\,(C_r(x_0), C_r(x_1), ..., C_r(x_m))$
(x_0 is any chosen element of X_0 and the set X_0 is treated as $C_r(x_0)$)
$\Rightarrow$ $\alpha(x_0, x_1, ..., x_m)$
(Induction hypothesis)
$\Rightarrow$ $(\exists x \in I(x))\ \alpha(x, x_1, ..., x_m)$

Similarly, the induction $\varphi \Rightarrow m_r(\varphi)$ can be proved. Finally, the proof in the case 5^0 is similar too.

Consider now again the $<, \leq$-formula $\varphi(x_1, ..., x_m)$ assuming that $x_1, ..., x_m$ are all free variables of this formula. Denote by $S(\varphi)$ the set of all values of $(x_1, ..., x_m) \in I(x_1) \times ... \times I(x_m)$ for which the formula $\varphi(x_1, ..., x_m)$ is true. Generalizing Definitions 2.1 and 2.2 we now introduce the following double definition.

Definition 4.2. *Let $r \in \mathbb{N}$ be a given element and let $P_r = X_1 \times ... \times X_m$ be a Cartesian product of some r-cells X_i (with $X_i \in \mathcal{D}_r\,(I(x_i))$). Then:*

(i) *The product P_r is feasible in the sense of the formula φ if and only if the condition $m_r(\varphi)\,(X_1, ..., X_m)$ is satisfied.*

(ii) *The product P_r is a solutional product in the sense of the formula φ if and only if the condition $M_r(\varphi)\,(X_1, ..., X_m)$ is satisfied.*

Suppose that all bounded variables of the formula $\varphi(x_1, ..., x_m)$ are $y_1, ..., y_m$ $(m \geq 0)$. Let $r \in \mathbb{N}$ be any fixed number. Each of the sets $\mathcal{D}_r(I(y_i))$ $(i = 1, ..., m)$ is a finite set. Consequently the quanitifiers of the forms

$$(\forall Y_i \in \mathcal{D}_r\,(I(y_i)))\,, \quad (\exists Y_i \in \mathcal{D}_r\,(I(y_i)))$$

can be in a standard way reduced to the corresponding conjunction, disjunction respectively. Then having in mind the way how the formulas $m_r(\varphi)(X_1, ..., X_m)$, $M_r(\varphi)(X_1, ..., X_m)$ are constructed one may for them say

(4.5) *$m_r(\varphi)\,(X_1, ..., X_m)\,, M_r(\varphi)\,(X_1, ..., X_m)$ are equivalent to some $\wedge - \vee$ expressions whose basic parts are some inequalities of the form*

$$m(f)(P_r) < M(g)(Q_r), \quad m(f)(P_r) \leq M(g)(Q_r)$$
$$M(f)(P_r) < m(g)(Q_r), \quad M(f)(P_r) \leq m(g)(Q_r)$$

where P_r, Q_r are Cartesian products of some r-cells.

Example 4.2. *Let φ be the formula*

$$(\forall x \in [a, b])\ (\exists y \in [c, d])\ f(x) \leq g(y)$$

and let

$$\mathcal{D}_2[a, b] = \{A_1, A_2, A_3\}\,, \mathcal{D}_2[c, d] = \{C_1, C_2\}$$

Then $m_2(\varphi)$ reads

$$(\forall X \in \mathcal{D}_2[a, b])\ (\exists Y \in \mathcal{D}_2[c, d])\ m(f)(X) < M(g)(Y)$$

which is logically equivalent to

(*) $\quad (m(f)(A_1) < M(g)(C_1) \ \vee \ m(f)(A_1) < M(g)(C_2))$

$$\wedge\ (m(f)(A_2) < M(g)(C_1) \ \vee \ m(f)(A_2) < M(g)(C_2))$$
$$\wedge\ (m(f)(A_3) < M(g)(C_1) \ \vee \ m(f)(A_3) < M(g)(C_2))$$

According to fact (4.5) *for any fixed $r \in \mathbb{N}$ and any product P_r one can in finite number of steps examine whether P_r is feasible or solutional product.*

There is one very interesting connection between the notions being feasible, and being solutional product P_r. These notions **are φ-dual to each other** which means that generally the following equivalence is true

(4.6) $\qquad P_r$ is $\neg\varphi$-*feasible*[1] $\Leftrightarrow P_r$ is not φ – *solutional*

The proof is simple:

P_r is $\neg\varphi$ -feasible

$\Leftrightarrow \quad m_r(\neg\varphi)\,(X_1,...,X_m)$ (where $P_r = X_1 \times ... \times X_m$)
By Definition 4.2

$\Leftrightarrow \quad \neg M_r(\varphi)\,(X_1,...,X_m)$
Since generally $m_r(\neg\varphi) = \neg M_r(\varphi)$

$\Leftrightarrow \quad P_r$ is not φ-solutional

Denote, like we did in section 2, by $S_r(\varphi)$, $F_r(\varphi)$ the unions of all solutional, feasible products P_r, with fixed $r \in \mathbb{N}$, respectively. Notice further that double implication (4.4) can be rewriten as follows

$$(\forall r \in \mathbb{N})\,(M_r(\varphi)(C_r(x_1),...,C_r(x_m)) \Rightarrow \varphi(x_1,...,x_m) \Rightarrow m_r(\varphi)\,(C_r(x_1),...,C_r(x_m)))$$

and also in this way[2]

(4.7) $$(\exists r \in \mathbb{N})\, M_r(\varphi)\,(C_r(x_1),...,C_r(x_m)) \Rightarrow \varphi(x_1,...,x_m) \Rightarrow (\forall r \in \mathbb{N})\, m_r(\varphi)\,(C_r(x_1),...,C_r(x_m))$$

Combining this double implication and Definiton 4.2 one immediately obtains the following natural assertion.

Theorem 4.2. *In general, the double inclusion*

(4.8) $$\bigcup_{r\in\mathbb{N}} S_r(\varphi) \subseteq S(\varphi) \subseteq \bigcap_{r\in\mathbb{N}} F_r(\varphi)$$

[1] $\neg\varphi$-feasible means "feasible in the sense of the formula $\neg\varphi$". A similar shorter way of writing is used on the right-hand side of (4.6) too.

[2] Using the standard properties of quantifiers as:

$$(\forall r \in \mathbb{N})\,(\psi_r \Rightarrow \varphi) \Leftrightarrow ((\exists r \in \mathbb{N})\,\psi_r \Rightarrow \varphi)$$

(providing that r is not a free variable of φ)
where ψ_r is $M_r\,(C_r(x_1),...,C_r(x_m))$.

is true.
About the sets $\bigcup_{r\in\mathbb{N}} S_r(\varphi)$, $\bigcap_{r\in\mathbb{N}} F_r(\varphi)$ now we point out that according to the well-known facts they are an open, a closed set respectively.

3. Considering double implication (4.7) one can put a natural question when one of the symbols $\Rightarrow$ may be replaced by the symbol $\Leftrightarrow$. The answer will be given by Theorem 4.3.
A formula φ is the so-called $\leq$-positive formula if it is built up using the relational symbol $\leq$ and the logical symbols $\wedge$, $\vee$, $\forall$, $\exists$ (i.e. without the negation symbol). In a quite similar way the notion of $<$-positive formula is introduced.

In order to prove Theorem 4.3 we shall use Lemma 4.1 below. In this lemma we use the following denotation:

If Δ is any one-dimensional segment then $\mathcal{D}'_r(\Delta)$ is the set of all one-dimensional r-cells X having at least one common point with Δ

Lemma 4.1. *Let ϕ be a positive $<$-formula whose all free variables are among $x_1,...,x_m$. Suppose that these variables have some initial values (from their intervals) and that there exists $k \in \mathbb{N}$ such that*

(*1) $\quad M_k(\phi)(C_k(x_1),...,C_k(x_m))$ *is true.*

Then there exists $r_0 \in \mathbb{N}$ such that for every $r \geq r_0$

$$(\forall X1 \in \mathcal{D}'_r(C_k(x_1)))...(\forall Xm \in \mathcal{D}'_r(C_k(x_m)))\ M_r(\phi)(X1,...,Xm)$$

is also true.

Proof. Denote by $l(\phi)$ the number of all logical symbols $\wedge,\vee,\forall,\exists$ occuring in the formula ϕ. The proof is by induction on the $l(\phi)$.

Case $l(\phi) = 0$. Then ϕ has a form $f(y_1,...,y_p) < g(z_1,...,z_q)$ where $\{y_1,...,y_p,z_1,...,z_q\} = \{x_1,...,x_m\}$. By hypothesis (*1) we have the following inequality

$$M(f)(C_k(y_1) \times ... \times C_k(y_p)) < m(g)(C_k(z_1) \times ... \times C_k(z_q))$$

where $k \in \mathbb{N}$ is a constant. According to the fact that generally

$$M(f)(Y_1 \times ... \times Y_p) - m(f)(Y_1 \times ... \times Y_p) \to 0$$
$$M(g)(Z_1 \times ... \times Z_q) - m(g)(Z_1 \times ... \times Z_q) \to 0$$
$(Y_i \in \mathcal{D}_r(I(y_i)), Z_j \in \mathcal{D}_r(I(z_j))$ are any one-dimensional r-cells)

when r tends to ∞, there exists a $r_0 \in \mathbb{N}$ such that for every $r \geq r_0$ (with $r \in \mathbb{N}$) we have the inequalities

(*2) $$M_r(f)(Y_1 \times ... \times Y_p) - m_r(f)(Y_1 \times ... \times Y_p) < \tfrac{1}{3}(m(g)(C_k(z_1) \times ... \times C_k(z_q)) - M(f)(C_k(y_1) \times ... \times C_k(y_p)))$$

(*3) $$M_r(g)(Z_1 \times ... \times Z_q) - m_r(g)(Z_1 \times ... \times Z_q) < \tfrac{1}{3}(m(g)(C_k(z_1) \times ... \times C_k(z_q)) - M(f)(C_k(y_1) \times ... \times C_k(y_p)))$$

(where $Y_i \in \mathcal{D}_r(I(y_i))$, $Z_j \in \mathcal{D}_r(I(z_j))$)

Let $Y_i \in \mathcal{D}_r(I(y_i))$, $Z_j \in \mathcal{D}_r(I(z_j))$ be any r-cells with the property:

There exist points y_i', z_j' such that

$y_i' \in Y_i \cap C_k(y_i)$, $z_j' \in Z_j \cap C_k(z_j)$ $\qquad (i = 1, ..., p; j = 1, ..., q)$

For any such $Y_1, ..., Y_p, Z_1, ..., Z_q$ we have the following argument:

$$\begin{aligned}
&m(g)(Z_1 \times ... \times Z_q) - M(f)(Y_1 \times ... \times Y_p)\\
&> M(g)(Z_1 \times ... \times Z_q) - m(f)(Z_1 \times ... \times Z_q)\\
&\quad -\tfrac{2}{3}\,(m(g)(C_k(z_1) \times ... \times C_k(z_q)) - M(f)(C_k(y_1) \times ... \times C_k(y_p)))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (*2), (*3))}\\
&\geq g(z_1', ..., z_q') - f(y_1', ..., y_p')\\
&\quad -\tfrac{2}{3}\,(m(g)(C_k(z_1) \times ... \times C_k(z_q)) - M(f)(C_k(y_1) \times ... \times C_k(y_p)))\\
&\geq m(g)(C_k(z_1) \times ... \times C_k(z_q)) - M(f)(C_k(y_1) \times ... \times C_k(y_p))\\
&\quad -\tfrac{2}{3}\,(m(g)(C_k(z_1) \times ... \times C_k(z_q)) - M(f)(C_k(y_1) \times ... \times C_k(y_p)))\\
&> 0
\end{aligned}$$

which completes the proof in Case $l(\phi) = 0$.

Case $l(\varphi) > 0$. The formula φ can have one of the forms

$1^0 \alpha \wedge \beta$, $2^0 \alpha \vee \beta$, $3^0\ (\exists v \in I(v))\, \alpha(v, x_1, ..., x_m)$ $4^0\ (\forall v \in I(v))\, \alpha(v, x_1, ..., x_m)$

Case 1^0. By hypothesis (*1) we have the following assumption

$M_k(\alpha \wedge \beta)(C_k(x_1), ..., C_k(x_m))$ is true.

Further we have the following argument:

$M_k(\alpha \wedge \beta)(C_k(x_1), ..., C_k(x_m))$

$\longrightarrow M_k(\alpha)(C_k(x_1), ..., C_k(x_m))$ and $M_k(\beta)(C_k(x_1), ..., C_k(x_m))$

$\longrightarrow$ There exists $r_1 \in \mathbb{N}$ such that for every $r \geq r_1$ and $(\forall X1 \in \mathcal{D}_r'(C_k(x_1)))...(\forall Xm \in \mathcal{D}_r'(C_k(x_m)))$ $M_k(\alpha)(X_1, ..., X_m)$ holds

and there exists $r_2 \in \mathbb{N}$ such that for every $r \geq r_2$ and $(\forall X1 \in \mathcal{D}_r'(C_k(x_1)))...(\forall Xm \in \mathcal{D}_r'(C_k(x_m)))$ $M_k(\beta)(X_1, ..., X_m)$ holds. (Using induction hypothesis)

$\longrightarrow$ for every $r \geq max(r_1, r_2)$ and $(\forall X1 \in \mathcal{D}_r'(C_k(x_1)))...(\forall Xm \in \mathcal{D}_r'(C_k(x_m)))$ $M_k(\alpha \wedge \beta)(X_1, ..., X_m)$ holds.

which completes the proof in case 1^0.

Case 2^0. By hypothesis (*1) we have the following assumption

$M_k(\alpha \vee \beta)(C_k(x_1), ..., C_k(x_m))$ is true, where $k \in \mathbb{N}$ is a constant.

Hence we conclude that

$M_k(\alpha)(C_k(x_1), ..., C_k(x_m))$ is true or $M_k(\beta)(C_k(x_1), ..., C_k(x_m))$ is true.

Let $M_k(\alpha)(C_k(x_1), ..., C_k(x_m))$ be true.

Further we have the following argument:

$M_k(\alpha)(C_k(x_1), ..., C_k(x_m))$

$\longrightarrow$ There exists $r_1 \in \mathbb{N}$ such that for every $r \geq r_1$ and $(\forall X1 \in \mathcal{D}_r'(C_k(x_1)))...(\forall Xm \in \mathcal{D}_r'(C_k(x_m)))$ $M_k(\alpha)(X_1, ..., X_m)$ holds. (Using induction hypothesis)

$\longrightarrow$ for every $r \geq r_1$ and $(\forall X1 \in \mathcal{D}_r'(C_k(x_1)))...(\forall Xm \in \mathcal{D}_r'(C_k(x_m)))$ $M_k(\alpha \vee \beta)(X_1, ..., X_m)$ holds (Using part (iii) of Definition 4.1)

which completes the proof in case 2^0, since the case when

$M_k(\beta)(C_k(x_1), ..., C_k(x_m))$ is true

can be treated in a quite similar way.

Case 3^0. By hypothesis (*1) we have the following assumption

$(\exists V \in \mathcal{D}_k(I(v)))M_k(\alpha)(V, C_k(x_1), ..., C_k(x_m))$ is true, where $k \in \mathbb{N}$ is a constant.

Hence we conclude that for some $W \in \mathcal{D}_k(I(v))$

(*4) $\qquad M_k(\alpha)(W, C_k(x_1), ..., C_k(x_m))$ is true.

This W can be treated as $C_k(v)$, where for the variable v one can take any initial value from the set W. Of course, we do not need to replace W by $C_k(v)$.

In connection with (*4) consider the formula $\alpha(v, x_1, ..., x_m)$ whose free variables are among $v, x_1, ..., x_m$. For this formula we have $l(\alpha) < l(\phi)$, consequently we may use induction hypothesis. Bearing in mind (*4) we obtain the following conclusion:

(*5) There exists $r_1 \in \mathbb{N}$ such that for every $r \geq r_1$ and $(\forall V \in \mathcal{D}_r'(W))(\forall X1 \in \mathcal{D}_r'(C_k(x_1)))...(\forall Xm \in \mathcal{D}_r'(C_k(x_m)))$ $M_k(\alpha)(V, X1, ..., Xm)$ holds.

To complete the proof in case 3^0 we shall prove:

For every $r \geq r_1$ and $(\forall X1 \in \mathcal{D}_r'(C_k(x_1)))...(\forall Xm \in \mathcal{D}_r'(C_k(x_m)))$.

(*6) $\qquad (\exists V \in \mathcal{D}_r(I(v)))M_r(\alpha)(V, X1, ..., Xm)$ holds.

Indeed, let $r \geq r_1$ be any fixed natural number and let X1,...,Xm be any elements of the sets $\mathcal{D}_r'(C_k(x_1)), ..., \mathcal{D}_r'(C_k(x_m))$ respectively. To prove (*6) it suffices to find 'a witness' for V. According to (*5) as a witness one may take any element of the set $\mathcal{D}_r'(W)$. So,the proof in case 3^0 is complete.

Case 4^0. By hypothesis (*1) we have the following assumption

(*7) $(\forall V \in \mathcal{D}_k(I(v)))M_k(\alpha)(V, C_k(x_1), ..., C_k(x_m))$ is true, where $k \in \mathbb{N}$ is a constant.

The set $\mathcal{D}_k(I(v))$ is a finite set. Let

$$S_1, S_2, ..., S_{kk}$$

be all of its elements. We point out that kk is a fixed element of the set $\mathbb{N}$. According to (*7) we have the following consequences

(*8) $M_k(\alpha)(S_1, C_k(x_1), ..., C_k(x_m))$

.
.
.

$M_k(\alpha)(S_{kk}, C_k(x_1), ..., C_k(x_m))$

are true.

In connection with it consider the formula $\alpha(v, x_1, ..., x_m)$ whose free variables are among $v, x_1, ..., x_m$. For this formula we have $l(\alpha) < l(\phi)$, consequently we may use induction hypothesis. Exactly said we shall use induction hypothesis kk times. Namely, in connection with (*8) we have the following conclusions

(*9) There exists $r_1 \in \mathbb{N}$ such that for every $r \geq r_1$
and $(\forall V \in \mathcal{D}'_r(S_1))(\forall X1 \in \mathcal{D}'_r(C_k(x_1)))...(\forall Xm \in \mathcal{D}'_r(C_k(x_m)))$
$M_k(\alpha)(V, X1, ..., Xm)$ holds.

.
.
.

There exists $r_{kk} \in \mathbb{N}$ such that for every $r \geq r_{kk}$
and $(\forall V \in \mathcal{D}'_r(S_{kk}))(\forall X1 \in \mathcal{D}'_r(C_k(x_1)))...(\forall Xm \in \mathcal{D}'_r(C_k(x_m)))$
$M_k(\alpha)(V, X1, ..., Xm)$ holds.

Let $r_0 = max(r_1, ..., r_{kk})$. Then according to (*9)

For every $r \geq r_0$ we have:

(*10) $(\forall V \in \mathcal{D}'_r(S_1))(\forall X1 \in \mathcal{D}'_r(C_k(x_1)))...(\forall Xm \in \mathcal{D}'_r(C_k(x_m)))$
$M_k(\alpha)(V, X1, ..., Xm)$ holds.

.
.
.

$(\forall V \in \mathcal{D}'_r(S_{kk}))(\forall X1 \in \mathcal{D}'_r(C_k(x_1)))...(\forall Xm \in \mathcal{D}'_r(C_k(x_m)))$
$M_k(\alpha)(V, X1, ..., Xm)$ holds.

To finish the proof in case 4^0 we shall prove:

For every $r \geq r_0$ and $(\forall X1 \in \mathcal{D}'_r(C_k(x_1)))...(\forall Xm \in \mathcal{D}'_r(C_k(x_m)))$
$(\forall V \in \mathcal{D}_r(I(v)))M_r(\alpha)(V, X1, ..., Xm)$ holds.

Indeed, let $r \geq r_0$ be any fixed natural number, further let X1,...,Xm be any elements of the sets $\mathcal{D}'_r(C_k(x_1)), ..., \mathcal{D}'_r(C_k(x_m))$ respectively, and finally let V be any element of the set $\mathcal{D}_r(I(v))$. So, we should prove that

(*11) $M_r(\alpha)(V, X1, ..., Xm)$ holds.

First, since:

$$S_1 \cup ... \cup S_{kk} = I(v)$$

the V must have a common element with some S_i, where $1 \leq i \leq kk$. Then using the i-th formula in (*10) we conclude that (*11) is true. Thus the proof in case 4^0 is complete, which means that Lemma 4.1 is also proved.

Now we go to prove Theorem 4.3.

Theorem 4.3. *Let all free variables of the formula φ be among the variables $x_1, ..., x_m$ (with $n \geq 0$). Then:*

(j) *If φ is a $<$-positive formula then the equivalence*

$$\varphi(x_1, ..., x_m) \Leftrightarrow (\exists r \in \mathbb{N})\, M_r(\varphi)\,(C_r(x_1), ..., C_r(x_m)) \tag{4.9}$$

holds.

(jj) *If φ is a $\leq$-positive formula then the equivalence*

$$\varphi(x_1, ..., x_m) \Leftrightarrow (\forall r \in \mathbb{N})\, m_r(\varphi)\,(C_r(x_1), ..., C_r(x_m)) \tag{4.10}$$

holds.

In both cases it is supposed that the variables $x_1, ..., x_m$ have any values from their segments $I(x_1), ..., I(x_m)$ respectively.

Proof. (j) Since the $\Leftarrow$-part of equivalence (4.9) is already proved (see (4.4)) we have prove the $\Rightarrow$-part, i.e. the implication

$$\varphi(x_1, ..., x_m) \Rightarrow (\exists r \in \mathbb{N})\, M_r(\varphi)\,(C_r(x_1), ..., C_r(x_m)) \tag{4.11}$$

As a matter of fact we are going to prove the following implication

$$\varphi(x_1, ..., x_m) \Rightarrow (\exists r_0 \in \mathbb{N})\,(\forall r \geq r_0)\, M_r(\varphi)\,(C_r(x_1), ..., C_r(x_m)) \tag{4.12}$$

which is a little more general than (4.11). Namely, if we take $r = r_0$ then from (4.12) it follows (4.11).

Denote by $l(\varphi)$ the number of all logical symbols $\wedge, \vee, \forall, \exists$ occuring in the formula φ. The proof is by induction on the $l(\varphi)$.
C a s e $l(\varphi) = 0$. Implication (4.12) reduces to some implication of the form

(*1) $f(y_1, ..., y_p) < g(z_1, ..., z_q)$
$\Rightarrow (\exists r_0 \in \mathbb{N})\,(\forall r \geq r_0)$
$M(f)\,(C_r(y_1) \times ... \times C_r(y_p)) < m(g)\,(C_r(z_1) \times ... \times C_r(z_q))$

where y_i, z_j are some of $x_1, ..., x_m$. Let[1] $\{y_1, ..., y_p, z_1, ..., z_q\} = \{x_1, ..., x_m\}$, where $m \leq n$. Let $X_0 = (x_1, ..., x_m) \in D$, where $D = I(x_1) \times ... \times I(x_m)$, be any point and let inequality

$$f(y_1, ..., y_p) < g(z_1, ..., z_q)$$

hold at this point. Denote $(y_1, ..., y_p)$, $(z_1, ..., z_q)$ by Y_0, Z_0 respectively. Let a, b be two real numbers which at the point X_0 satisfy the inequality[1]

(*2) $$f(Y_0)) < a < b < g(Z_0)$$

In virtue of axiom (0.2) it follows that there exists a positive number d such that the following inequalities

(*3) $$M(f)(\Delta_1) - m(f)(\Delta_1) < \frac{b-a}{2}, \quad M(g)(\Delta_2) - m(g)(\Delta_2) < \frac{b-a}{2}$$

are satisfied whenever

(*4) Δ_1, Δ_2 are any $p-, q-$dimensional subsegment of $I(y_1) \times ... \times I(y_p)$, $I(z_1) \times ... \times I(z_q)$ respectively, $Y_0 \in \Delta_1$, $Z_0 \in \Delta_2$ and

$$diam\, \Delta_1 < d, diam\, \Delta_2 < d$$

From (*3) one can easily prove the following inequality

(*5) $$M(f)(\Delta_1) < m(g)(\Delta_2)$$

under the condition (*4). Next, in virtue of Definition 1.2 and condition (1.9)(v) it follows that there exists $r_0 \in \mathbb{N}$ such that for any $r \geq r_0$ the inequalities

$$diam\,(C_r(y_1) \times ... \times C_r(y_p)) < d, \quad diam\,(C_r(z_1) \times ... \times C_r(z_q)) < d$$

hold. Applying (*5) we get the inequality

$$M(f)\,(C_r(y_1) \times ... \times C_r(y_p)) < m(g)\,(C_r(z_1) \times ... \times C_r(z_q))$$

which completes the proof in case $l(\varphi) = 0$.

C a s e $l(\varphi) > 0$. The formula φ can have one of the forms

$$1^0 \alpha \wedge \beta,\ 2^0 \alpha \vee \beta,\ 3^0\,(\exists v \in I(v))\,\alpha(v, x_1, ..., x_m)\ 4^0\,(\forall v \in I(v))\,\alpha(v, x_1, ..., x_m)$$

Case 1^0. By the induction hypothesis we have

$$\alpha(x_1, ..., x_m) \Rightarrow M_r(\alpha)\,(C_r(x_1), ..., C_r(x_m)) \quad \text{for } r \geq r'$$
$$\beta(x_1, ..., x_m) \Rightarrow M_r(\alpha)\,(C_r(x_1), ..., C_r(x_m)) \quad \text{for } r \geq r''$$

where r', r'' are some elements of $\mathbb{N}$. Taking $r_0 = \max(r', r'')$, then for every $r \geq r_0$ we have the following implication argument

[1]Of course this is only a technical assumption.

[1]Instead of $f(y_1, ..., y_p), g(z_1, ..., z_q)$ we wrote $f(Y_0), g(Z_0)$ respectively.

$\alpha(x_1, ..., x_m) \wedge \beta(x_1, ..., x_m)$

$\Rightarrow$ $M_r(\alpha)\,(C_r(x_1), ..., C_r(x_m)) \wedge M_r(\beta)\,(C_r(x_1), ..., C_r(x_m))$

$\Rightarrow$ $M_r(\alpha \wedge \beta)\,(C_r(x_1), ..., C_r(x_m))$
By Definition 4.1, part (ii)

which completes the proof in case 1^0.

Proof in case 2^0 is similar. In case 3^0 we have the following implication argument

$(\exists v \in I(v))\,\alpha(v, x_1, ..., x_m)$

$\Rightarrow$ $\alpha(v_0, x_1, ..., x_m)$ ($v_0 \in I(v_0)$ is some fixed element)

$\Rightarrow$ $(\exists r_0 \in \mathbb{N})\,(\forall r \geq r_0)\,M_r(\alpha)\,(C_r(v_0), C_r(x_1), ..., C_r(x_m))$ (Induction hypothesis)

$\Rightarrow$ $(\exists r_0 \in \mathbb{N})\,(\forall r \geq r_0)\,(\exists V \in \mathcal{D}_r\,(I(v)))\,M_r(\alpha)\,(V, C_r(x_1), ..., C_r(x_m))$

which completes the proof in case 3^0.

In case 4^0 we have to prove the following implication

$(\forall v \in I(v))\,\alpha(v, x_1, ..., x_m)$
$\Rightarrow (\exists r_0 \in \mathbb{N})\,(\forall r \geq r_0)\,(\forall V \in \mathcal{D}_r\,(I(v)))\,M_r(\alpha)\,(V, C_r(x_1), ..., C_r(X_m))$

Suppose the contrary. So we have the following assumptions[1]

(*) $$(\forall v \in I(v))\,\alpha(v, x_1, ..., x_m)$$

(**) $$\neg\,(\exists r_0 \in \mathbb{N})\,(\forall r \geq r_0)\,(\forall V \in \mathcal{D}_r\,(I(v)))\,M_r(\alpha)\,(V, C_r(x_1), ..., C_r(x_m))$$

From (**) it follows

$$(\forall r_0 \in \mathbb{N})\,(\exists r \geq r_0)\,(\exists V \in \mathcal{D}_r\,(I(v)))\,\neg M_r(\alpha)\,(V, C_r(x_1), ..., C_r(x_m))$$

hence we conclude that there exist[1] a sequence $r_1 < r_2 < r_3 < ...$ and a number $v_0 \in I(v)$ such that $C_{r_1}(v_0), C_{r_2}(v_0), ...$ satisfy the conditions

(Δ) $$\neg M_{r_i}\,(C_{r_i}(v_0), C_{r_i}(x_1), ..., C_{r_i}(x_m)) \quad (i = 1, 2, ...)$$

Now from (*) it follows $\alpha(v_0, x_1, ..., x_m)$. Hence by induction hypothesis we conclude

($\Delta\Delta$) $$(\exists r_0)\,(\forall r \geq r_0)\,M_r\,(C_r(v_0), C_r(x_1), ..., C_r(x_m))$$

Bearing in mind Lemma 4.1 we see that (Δ) and ($\Delta\Delta$) contradict each other. So, the proof of (j) is complete.

(jj) For the moment denote equivalence (4.9) by $P(\varphi) \Leftrightarrow Q(\varphi)$. Replacing φ by $\neg\varphi$, where this new φ is any $\leq$-positive formula, we obtain the equivalence

$$P(\neg\varphi) \Leftrightarrow Q(\neg\varphi)$$

[1]During the proof we suppose that $x_1, ..., x_m$ have any fixed values.

[1]For instance, applying Axiom of Choice and Cantor's theorem on sequences of nested segments.

Further, by negating both sides in this equivalence we obtain the equivalence

$$\neg P(\neg\varphi) \Leftrightarrow \neg Q(\neg\varphi)$$

After a short logical calculation this equivalence reduces to (4.10). In other words equivalences (4.9) and (4.10) are φ-dual to each other.
The proof of Theorem 4.3 is complete.

Now we give some examples of equivalences of the types (4.9) and (4.10)

$$f(x) > g(y) \Leftrightarrow (\exists r \in \mathbb{N})\ m(f)(C_r(x)) > M(g)(C_r(y))$$

$$f(x,y) > g(y) \Leftrightarrow (\exists r \in \mathbb{N})\ m(f)(C_r(x) \times C_r(y)) > M(g)(C_r(y))$$

$$(\forall x \in I(x))\,(\exists y \in I(y))\, f(x) > g(y)$$
$$\Leftrightarrow (\exists r \in \mathbb{N})\,(\forall X \in \mathcal{D}_r(I(x)))\,(\exists Y \in \mathcal{D}_r(I(y)))\, m(f)(X) > M(g)(Y)$$

$$f(x) = 0 \Leftrightarrow (\forall r \in \mathbb{N})\, m(f)\,(C_r(x)) \leq 0 \leq M(f)\,(C_r(x))$$
$$(\text{For: } f(x) = 0 \Leftrightarrow f(x) \leq 0 \land f(x) \geq 0)$$

Double inclusion (4.8) may be treated as a set-theoretical interpretation of double implication (4.4), Similarly to this there is the corresponding set-theoretical interpretation of Theorem 4.3. Namely we have the following assertion

Theorem 4.4.

Let $x_1, ..., x_m$ be all the free variables of the formula φ. Then:

(j) If φ is a $<$-positive formula then the equality

(4.13) $$S(\varphi) = \bigcup_{r \in \mathbb{N}} S_r(\varphi)$$

holds.

(jj) If φ is a $\leq$-positive formula then the equality

(4.14) $$S(\varphi) = \bigcap_{r \in \mathbb{N}} F_r(\varphi)$$

holds.

Proof. (jj) The $\subseteq$-part is covered by Theorem 4.2. To prove $\supseteq$-part suppose that $(x_1, ..., x_m)$ is any element of the $\bigcap F_r$. According to Definition 4.2, part (i) we conclude the following

(*) $$m_r(\varphi)\,(C_r(x_1), ..., C_r(x_m)), \quad \text{for every } r \in \mathbb{N}$$

To prove $\varphi(x_1, ..., x_m)$ we suppose the contrary

$$\neg\varphi(x_1, ..., x_m)$$

Since $\neg\varphi$ is a $<$-positive formula then applying (4.9) we conclude

$$M_{r_0}(\neg\varphi)\,(C_{r_0}(x_1), ..., C_{r_0}(x_m)), \quad \text{for some } r_0 \in \mathbb{N}$$

Using Definition 4.1, part (iv) we conclude

$$\neg m_{r_0}(\varphi)\,(C_{r_0}(x_1), ..., C_{r_0}(x_m)), \quad \text{for some } r_0 \in \mathbb{N}$$

The proof is completed since this contradicts to (*).
(j) Equality (4.13) can be easily proved using equality (4.14) and general equivalence (4.6).

5. SOLVING A FIRST ORDER $<, \leq$-FORMULA

In this section we state various applications of the results established in section 4, such as:

- Problem of constrained optimization (Problem 5.2, Problem 5.3)
- Problem of unconstrained optimization (Problem 5.1)
- min-max problems (Problem 5.4)
- Problems from Interval Mathematics (Problem 5.5)

1. Let φ be a given positive $<, \leq$-formula whose all variables, free or bounded, are $v_1, ..., v_t$. Concerning their segments $I(v_i)$ $(i = 1, ..., t)$ we suppose that for each of them one cell-decomposition $\mathcal{D}(I(v_i))$ is chosen (see Definition 1.2). We consider the following classs of problems.

Class 5.1

(i) *If $x_1, ..., x_n$ are all the free variables of the formula φ, find[1] all values of $x_i \in I(x_i)$ $(i = 1, ..., n)$ for which the formula φ is satisfied.*

(ii) If the formula φ has no free variables establish whether φ is true or false.

Notice that the problems treated in section 2 belong to this class. Namely, it suffices to see that any inequality system (see 2.1)

$$f_1 \geq 0, ..., f_k \geq 0$$

may be treated as the conjunction:

$$f_1 \geq 0 \land ... \land f_k \geq 0$$

which is a positive $\leq$-formula.
Consider first the problem of type Class 5.1, case (i). If φ is a positive $\leq$-formula then due to Theorem 4.4 for the set $S(\varphi)$ of all its solutions we have the equality

$$S(\varphi) = \textstyle\bigcap_{i \in \mathbb{N}} F_r(\varphi)$$

where Definition 4.2 also understood. Consequently:

(5.1) *To find the set $S(\varphi)$ one can employ a procedure almost identical to that described by text[2] (2.3). More precisely, like in (2.3), instead of the sequence $\langle F_r(\varphi)\rangle$ one can use the sequence $\langle F'_r(\varphi)\rangle$ for which we also have the equality*

$$S(\varphi) = \bigcap_{r \in \mathbb{N}} F'_r(\varphi)$$

and additionally $\langle F'_r(\varphi)\rangle$ is a monotone sequence.

---

[1] In other words, solve φ in $x_1, ..., x_n$.
[2] The following conventions

$$[a_i, b_i] = I(x_i), \quad i = 1, ..., n; \quad D = [a_1, b_1] \times ... \times [a_n, b_n]$$

are adopted.

The solving procedure, as we saw in section 2, can be profoundly improved by using the notions of **solutional** (see Definition 4.2) and **indetermined** products (a product P_r is indetermined if and only if P_r is a feasible but not a solutional product). In more detail the improvement is described by text (2.6). However, unlike the problem treated in section 2 now some additional difficulties can arise in connection with the fact that φ can have some quantifiers. Such questions will be discussed in Problems 5.1, 5.2, 5.3 and particulary in part 2 of this section.

If φ is a positive $<$-formula then the set $S(\varphi)$ can be determined using the equality (see Theorem 4.4)

$$S(\varphi) = \bigcup_{r \in \mathbb{N}} S_r(\varphi)$$

Notice also that $S(\varphi)$ can be determined by means of the equality

$$S(\varphi) = \Delta \setminus S(\neg\varphi)$$

provided that $S(\neg\varphi)$ is determined by the procedure (5.1) described above[3)]

Finally, let φ be a positive $<, \leq$-formula containing both of the symbols $<, \leq$. Denote φ by $\varphi(<, \leq)$ also. Replacing the symbol $<$ by the symbol $\leq$ from the formula $\varphi(<, \leq)$ one obtains the formula $\varphi(\leq, \leq)$. Similarly, $\varphi(<, <)$ is the formula obtained from $\varphi(<, \leq)$ by replacing the symbol $\leq$ by the symbol $<$. Now we emphasize the following double implication

$$(*1) \qquad \varphi(<, <) \Rightarrow \varphi(<, \leq) \Rightarrow \varphi(\leq, \leq)$$

which can be easily proved. From (*1) it follows the following double inclusion

$$(*2) \qquad S\,(\varphi(<, <)) \subseteq S\,(\varphi(<, \leq)) \subseteq S\,(\varphi(\leq, \leq))$$

The sets $S\,(\varphi(<, <))$, $S\,(\varphi(\leq, \leq))$ can be determined by the procedures stated above. Accordingly, the set $S\,(\varphi(<, \leq))$ can be by (*2) approximatively determined.

Next we consider the problem of type Class 5.1, case (ii). Now the main role has the double implication of the form (4.4) and the fact that, in principle, for any fixed $r \in \mathbb{N}$ one can in a finite number of steps examine whether $m_r(\varphi)$, $M_r(\varphi)$ are satisfied (see (4.5)). Accordingly in order to solve a problem of type Class 5.1, case (ii) we employ the following procedure

(5.2) (i) *We start with $r = 0$.*
(ii) *We calculate $m_r(\varphi)$. Then:*
If $m_r(\varphi)$ is false the procedure halts and the answer is: φ is false. Otherwise, we go to (iii).
(iii) *We calculate $M_r(\varphi)$. Then*
If $M_r(\varphi)$ is true the procedure halts and the answer is: φ is true. Otherwise, we go to (iv).
(iv) *We replace r by $r+1$ and go to (ii).*

---

3) For $\neg\varphi$ is equivalent to some positive $\leq$-formula.

Concerning this procedure generally there are two possibilities

(5.3) (j) *It stops at some step r_0.*
(jj) *It never stops.*

In case (j) we are able to establish effectively whether the formula φ is true or false. In case (jj) according to equivalences (4.9), (4.10) (with $n = 0$) we have the following conclusions:

1^0 φ is false if φ is a positive $<$-formula.
2^0 φ is true if φ is a positive $\leq$-formula.

An example in which case (j) appears is when φ is the formula

$$(\exists x \in [1, 2])\, x * x \geq 2,$$

but if φ is the formula

$$(\exists x \in [1, 2])\, x * x = 2$$

we have case (jj).

2. Now we are going to state some problems belonging to Class 5.1.

Problem 5.1. *Let $f : D \to \mathbb{R}$ ($D = [a_1, b_1] \times \ldots \times [a_n, b_n]$) be a given m-M function. We seek all points $(x_1, \ldots, x_n) \in D$ at which this function attains the minimum value, i.e. we solve in $(x_1, \ldots, x_n) \in D$ the formula*

$$(5.4) \qquad (\forall y_1 \in [a_1, b_1]) \ldots (\forall y_n \in [a_n, b_n])\, f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$$

This is obviously a problem of Class 5.1 (i). The corresponding definition of feasible products (of r-cells) reads

(5.5) *A product $X_1 \times \ldots \times X_n$ of r-cells X_i is feasible (in the sense of (5.4)) if it satisfies the condition*

$$(*)\ (\forall Y_1 \in \mathcal{D}_r[a_1, b_1]) \ldots (\forall Y_n \in \mathcal{D}_r[a_n, b_n])$$
$$m(f)\,(X_1 \times \ldots \times X_n) \leq M(f)\,(Y_1 \times \ldots \times Y_n)$$

Denote by Min the set of all solutions of (5.4). This set is not empty. Next, it has the following property

(5.6) *If $P, Q \in$ Min then $f(P) = f(Q)$ (Uniqueness of the minimum value)*

Further, suppose that $D' \subseteq D$ is any non-empty set. Then obviously the following implication is valid

$$(5.7) \qquad (\forall (y_1, \ldots, y_n) \in D)\, f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$$
$$\Rightarrow (\forall (y_1, \ldots, y_n) \in D')\, f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$$

Combining (5.6) and (5.7) the following fact follows immediately

(5.8) *Let D', with Min $\subseteq D' \subseteq D$ be any set. Then the set of all solutions $(x_1, \ldots, x_n) \in D'$ of the formula*

$$(\forall (y_1, \ldots, y_n) \in D')\, f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$$

is just equal to Min.

This fact can be used in the following manner

(5.9) *In the solving procedure we step-by-step replace the initial domain* D *by the sets* $F'_1, ..., F'_r, ...$ *respectively.*

Practically that means that part (*) of Definition (5.5) is replaced by the following one

(*') $\forall (Y_1, ..., Y_n) \in \left((\mathcal{D}_r[a_1, b_1] \times ... \times \mathcal{D}_r[a_n, b_n]) \bigcap F'_{r-1}\right)$

$$m(f)\,(X_1 \times ... \times X_n) \leq M(f)\,(Y_1 \times ... \times Y_n)$$

In such a way the for-loops of the variables $Y_1, ..., Y_n$ are profoundly diminished.

The next question is about the number of the feasible cells in the r-th step, denoted say by $fis(r)$. At first, it is not difficult to see that this number depends on the equality of m-M pairs used in the solving procedure. For instance, consider the case in which:

$$D = [0, 2], \quad f(x) = x^2 - 2x + 1$$

Applying the ordinary binary tree, when in the r-th step any cell is of the form

$$\Delta = [l, d], \text{ with } l = i/2^{r-1}, d = l + 1/2^{r-1} \quad (i = 0, ..., 2^r - 1)$$

it can be easily seen that

1^0 If we use the formulas

$$m(f)(\Delta) = l^2 - 2d + 1, M(f)(\Delta) = d^2 - 2l + 1$$

then $fis(r) \sim 2^{(r-1)/2}$, when $r \to +\infty$.

2^0 But, by using the idea of ideal m-M pairs (see Lemma 1.1) it is not difficult to conclude that $fis(r) = 4$ $(r \geq 2)$.

Return now to Problem 5.1 in the general case assuming:

(5.10) *The function* f *has the first order partial derivatives* $\partial f/\partial x_1$, ..., $\partial f/\partial x_n$ *for all* $(x_1, ..., x_n) \in D$ *and these derivatives are m-M functions.*

Suppose also that

(Δ) *The function* f *attains its minimum at some point* $(c_1, ..., c_n) \in$ *Interior*(D)

According to the well known fact at the point $(c_1, ..., c_n)$ all equations

(5.11) $$\frac{\partial f}{\partial x_1} = 0, ... \frac{\partial f}{\partial x_n} = 0$$

must be satisfied. In conection with it to the feasibility criterium (5.7) (*') we may impose the following additional requirements

(5.12) $$m\,(f'_{x_i})\,(X_1 \times ... \times X_n) \leq 0 \leq M\,(f'_{x_i})\,(X_1 \times ... \times X_n) \quad (i = 1, ..., n)$$

Of course, in such a way the initial feasibility criterium is refined profoundly.

Now we consider Problem 5.1 under the condition (5.10) only (i.e. without (Δ)). In that case, as it is well known, roughly speaking we first correspond to the given function f several "subfunctions" $f_1, ..., f_k$ and after that we seek their minimums separately. Each of problems: find $\min(f_i)$ is either of type (Δ) or is trivial. Then for $\min(f)$ we use the following equality

$$\min(f) = \min(\min(f_1), ..., \min(f_k)).$$

To be more clear we illustrate this idea in cases $n = 1$, $n = 2$.

Case $n = 1$. Then $D = [a_1, b_1]$, $f : D \to \mathbb{R}$. The subfunctions are f_1, f_2, f_3:

$$f_1(x) = f(x) \text{ for } x \in (a_1, b_1); \quad f_2(x) = f(a_1); \quad f_3(x) = f(b_1)$$

First we seek $\min(f_1)$, when we also add the condition $f'(x) = 0$. It may happen that $\min(f_1)$ does not exist. For $\min(f)$ we have the equality

$$\min(f) = \min\,(\min(f_1), f(a_1), f(b_1))$$

if $\min(f_1)$ exists, otherwise

$$\min(f) = \min\,(f(a_1), f(b_1))$$

Case $n = 2$. Then $D = [a_1, b_1] \times [a_2, b_2]$. $f = D \to \mathbb{R}$. The subfunctions are $f_1, f_2, ..., f_9$ determined as follows:

$f_1(x, y) = f(x, y)$, $a_1 < x < b_1$, $a_2 < y < b_2$. The equalities $f_x = 0$, $f_y = 0$ may be added.

$f_2(y) = f(a_1, y)$, $a_2 < y < b_2$. The equality $\frac{\partial}{\partial y} f(a_1, y) = 0$ may be added.

$f_3(y) = f(b_1, y)$, $a_2 < y < b_2$. The equality $\frac{\partial}{\partial y} f(a_2, y) = 0$ may be added.

$f_4(x) = f(x, a_2)$, $a_1 < x < b_1$. The equality $\frac{\partial}{\partial x} f(x, a_2) = 0$ may be added.

$f_5(x) = f(x, b_2)$, $a_1 < x < b_1$. The equality $\frac{\partial}{\partial x} f(x, b_2) = 0$ may be added.

$f_6 = f(a_1, a_2)$, $f_7 = f(a_1, b_2)$, $f_8 = f(b_1, a_2)$, $f_9 = f(b_1, b_2)$

For $\min(f)$ we have the equality

$$\min(f) = \min\,(\min(f_1), \min(f_2), ..., \min(f_5), f_6, f_7, f_8, f_9)$$

If some $\min(f_i)$, where $i \leq 1 \leq 5$ does not exist then the previous equality does not contain the term $\min(f_i)$.

Remark 5.1. In m-M Calculus we usually use 'the cell-decompostion strategy'. But, we can use another strategy as well. Here we shall state a skech[4)] of a procedure LS by which under some conditions one can find a local minimum or a saddle point of the function f from Problem 5.1.

Let δ be some positive real number, choosen arbitrarily. If $(x_1, ..., x_n) \in D$ then by $\Delta(x_1, ..., x_n, \delta)$ we denote the Cartesian product $[x_1 - \delta, x_1 + \delta] \times ... \times [x_n - \delta, x_n + \delta]$. We suppose that $f : D \to \mathbb{R}$ satisfies the condition

[4)]A complete version will be published separately

Function f has the second order partial derivatives $f''_{x_1,x_1}, ..., f''_{x_n,x_n}$ *and these derivatives are m-M functions in each* $\Delta(x_1, ..., x_n, \delta)$ *where* $(x_1, ..., x_n) \in D$.

In the procedure LS we shall use the following general fact:

Let $g : [a-h, a+h] \to \mathbb{R}$ *be a function having the first order derivative* $g'(x)$ *for every* $x \in (a-h, a+h)$*, whose modulus* $|g'(x)|$ *is bounded by some positive constant* K. *If* $g(a) > 0$ *then* $g(x) > 0$ *for every* $x \in [a-h', a+h']$ *where* $h' = min(h,\ g'(a)/K)$

In procedure LS we use the following constants, chosen arbitrarily :

S_{max} - the maximum number of steps in the procedure
Mem $\in \{0, 1, ..., n\}$ - an auxiliary number.

Procedure LS (partly described in 'Pascal style') reads:

We start with an initial point $(p_1, ..., p_n)$ from D.

$k := 1$; For $i := 1$ to n do $x_i := p_i$;
100: *Mem*:= 0;

For $i := 1$ to n do

Begin
If $f'_{x_i}(x_1, ..., x_n) > 0$ then
Begin if $x_i = a_i$ then *Mem:=Mem +1* else
$x_i := x_i$-min$(\delta,\ f'_{x_i}(x_1, ..., x_n)/M(|f''_{x_i,x_i}|)(\Delta))$
End
else if $f'_{x_i}(x_1, ..., x_n) < 0$ then
Begin if $x_i = b_i$ then *Mem:=Mem +1* else
$x_i := x_i$+min$(\delta,\ f'_{x_i}(x_1, ..., x_n)/M(|f''_{x_i,x_i}|)(\Delta))$
End
else *Mem:=Mem+1*
End

If *Mem*= n then write(' Result is ',$x_1, ..., x_n$)
else if $k < S_{max}$ then Begin k:=k+1; goto 100 End
else write('Approximative result is',$x_1, ..., x_n$)

It is supposed that we use the following general equality:

$$M(|g|)(\Delta) = max(|m(g)(\Delta)|, |M(g)(\Delta)|), \quad \text{where } g \text{ is } f''_{x_i,x_i}.$$

Problem 5.2. *Let* $g, f_1, ..., f_k : D \to \mathbb{R}$*, where* $D = [a_1, b_1] \times ... \times [a_n, b_n]$ *be given m-M functions. Let A be the set of all points* $(x_1, ..., x_n) \in D$ *satisfying the inequalities*

(5.13) $$f_1(x_1, ..., x_n) \geq 0, ...f_k(x_1, ..., x_n) \geq 0$$

Restricting the function g to the set A we seek the set S of all points $(x_1, ..., x_n) \in A$ *at which g attains the minimum value* (**the problem of constrained optimization**[5]) *under the condition* $(x_1, ..., x_n) \in D$).

In other words we seek all points $(x_1, ..., x_n) \in D$ satisfying the following conditions

(5.14) $1^0 f_1(x_1, ..., x_n) \geq 0, ..., f_k(x_1, ..., x_n) \geq 0$

$$2^0\ (\forall (y_1, ..., y_n) \in D)\, [f_1(y_1, ..., y_n) \geq 0, ...f_k(y_1, ..., y_k) \geq 0 \Rightarrow g(x_1, ..., x_n) \leq g(y_1, ..., y_n)]$$

Obviously the sets A, S are connected by the equivalence

$$S \neq \emptyset \Leftrightarrow A \neq \emptyset$$

However, (5.14) treated as a conjunction of its parts 1^0 and 2^0 *is not* a $\leq$-positive formula, therefore we cannot apply a procedure like (5.1). In connection with this trouble we put the following assumption

(5.15)*If at some point* $(x_1, ..., x_n) \in D$ *the inequalities*

$$f_1(x_1, ..., x_n) \geq 0, ...f_k(x_1, ..., x_n) \geq 0$$

are satisfied then in each neighbourhood $N(x_1, ..., x_n)$ *of this point there is a point* $(x_{1o}, ..., x_{no}) \in D$ *satisfying the inequalities*

$$f_1(x_{1o}, ..., x_{no}) > 0, ...f_k(x_{1o}, ..., x_{no}) > 0$$

Using this assumption we first prove the following equivalence[6]

(5.16) $(\forall y_1) ... (\forall y_n)\, [(\forall i)\, f_i(y_1, ..., y_n) \geq 0 \Rightarrow g(x_1, ..., x_n) \leq g(y_1, ..., y_n)]$

is equivalent to

$$(\forall y_1) ... (\forall y_n)\, [(\forall i)\, f_i(y_1, ..., y_n) > 0 \Rightarrow g(x_1, ..., x_n) \leq g(y_1, ..., y_n)]$$

Indeed, $\Rightarrow$-part is trivial. To prove $\Leftarrow$-part suppose that $(x_1, ..., x_n)$ has any fixed value $(x_{1o}, ..., x_{no}) \in D$ and that the condition $(\forall i)\, f_i(y_1, ..., y_n) \geq 0$ is fulfilled at some point $(y_1, ..., y_n) \in D$. If $g(x_{1o}, ..., x_{no}) \leq g(y_1, ..., y_n)$ the proof is completed. Suppose now that the oposite inequality $g(x_{1o}, ..., x_{no}) > g(y_1, ..., y_n)$ is satisfied. Because of the continuity of the function g this inequality is also satisfied in the set $N(y_1, ..., y_n) \cap D$, where $N(y_1, ..., y_n)$ is some neighbourhood of the point $(y_1, ..., y_n)$. However, using (5.15) we conclude that in this set there is a point $(y_{1o}, ..., y_{no})$ at which the inequalities $(\forall i)\, f_i(y_{1o}, ..., y_{no}) > 0$ hold, and by the premise of the $\Leftarrow$-part we also have the inequality $g(x_{1o}, ..., x_{no}) \leq g(y_{1o}, ..., y_{no})$. The proof is completed, for we have obtained a contradiction.

We now show that under the condition (5.15) the considered problem of optimization can be reduced to the problem of Class 5.1. Indeed, we have the following chain of logical equivalences

[5] Usualy in the literature the set A is called *a feasible* set.
[6] Instead of $(\forall y_j \in [a_j, b_j])$, $(\forall i \in \{1, ..., k\})$ we write shortly $(\forall y_j)$, $(\forall i)$ respectively.

$$(\forall i)\, f_i(x_1,...,x_n)) \geq 0 \wedge (\forall y_1)\dots(\forall y_n)\,[(\forall i)\, f_i(y_1,...,y_n) \geq 0$$
$$\Rightarrow g(x_1,...,x_n) \leq g(y_1,...,y_n)]$$

(This is the formula which corresponds to the considered problem)

$$\Leftrightarrow (\forall i)\, f_i(x_1,...,x_n)) \geq 0 \wedge (\forall y_1)\dots(\forall y_n)\,[(\forall i)\, f_i(y_1,...,y_n) > 0$$
$$\Rightarrow g(x_1,...,x_n) \leq g(y_1,...,y_n)]$$

(Using (5.15))

$$\Leftrightarrow f_i(x_1,...,x_n) \geq 0 \wedge (\forall y_1)\dots(\forall y_n)\,[(\exists i)\, f_i(y_1,...,y_n) \leq 0$$
$$\vee g(x_1,...,x_n) \leq g(y_1,...,y_n)]$$

(For: $(p \Rightarrow q) \Leftrightarrow \neg p \vee q$, etc.)

The proof is completed, because we have obtained a positive $\leq$-formula. So, we can apply a procedure of type (5.1). Accordingly, the definition of feasible Cartesian products of r-cells reads:

(5.17) *A Cartesian product $X_1 \times ... \times X_n$ of r-cells X_i is feasible if it satisfies the following conditions*[7]

(i) $(\forall i \in \{1,...,k\})\, M(f_i)(X_1 \times ... \times X_n) \geq 0$

(ii) $(\forall Y_1 \in \mathcal{D}_r([a_1,b_1]))\dots(\forall Y_n \in \mathcal{D}_r([a_n,b_n]))$
$(\forall i \in \{1,...,k\})\, m(f_i)\,(Y_1 \times ... \times Y_n) > 0$
$\Rightarrow m(g)\,(X_1 \times ... \times X_n) \leq M(g)\,(Y_1 \times ... \times Y_n)$

In the spirit of the solving procedure if the set A is empty then at some step r it will happen[8] $fis(r) = 0$.

As in Problem 5.1 now we may also use the fact of the form (5.9). In other words in (5.17) we may replace the part

$$(\forall Y_1 \in \mathcal{D}_r([a_1,b_1]))\dots(\forall Y_n \in \mathcal{D}_r([a_n,b_n]))$$

by the following

$$\forall (Y_1,...,Y_n) \in \left(\mathcal{F}'_{r-1} \cap (\mathcal{D}_r([a_1,b_1] \times ... \times \mathcal{D}_r([a_n,b_n]))\right)$$

In such a way for-loops of the variables Y_i are profoundly diminished. Similarly as in Problem 5.1 under certain conditions we may add new requirements to the feasibility criterion. So, suppose that each of the functions $f_1,...,f_k,g$ satisfies a condition of type (5.10) and the function g attains the minimum value at some point, which is an internal point of the set A. Then to the conditions (5.14) 1^0, 2^0 we may add the following equations

(5.18) $$\frac{\partial g}{\partial x_1} = 0, \dots \frac{\partial g}{\partial x_n} = 0$$

[7]In the formulation of (ii) the tautology $(\neg p \vee q) \Leftrightarrow (p \Rightarrow q)$ is employed.

[8]Recall that $fis(r)$ is the number of all feasible r-cells.

Accordingly, the feasibility criterium should have also these requirements

$$m\left(\frac{\partial g}{\partial x_i}\right)(X_1 \times ... \times X_n) \leq 0 \leq M\left(\frac{\partial g}{\partial x_i}\right)(X_1 \times ... \times X_n)$$

Remark 5.2. If we replace (5.13) in Problem 5.2 by the following disjunction

$$f_1(x_1,...,x_n) \geq 0 \vee ... \vee f_k(x_1,...,x_n) \geq 0$$

we get a new problem which can be solved in a similar way as Problem 5.2 (such a problem belongs to the disjunctive-optimization problems).

Problem 5.3. *We get this problem from Problem 5.2 by replacing (5.13) by the following conditions*

(5.19) $f_1(x_1,...,x_n) = 0, ..., f_s(x_1,...,x_n) = 0,$
$f_{s+1}(x_1,...,x_n) \geq 0, ..., f_k(x_1,...,x_n) \geq 0 \qquad (s \geq 1,\ k \geq s)$

Now the set A is defined as the set of all points $(x_1,...,x_n) \in D$ satisfying (5.19). This problem is more complicated than Problem 5.2. In order to see this fact better suppose $s = 1, k = 2, n = 1$. So, briefly said we seek the values $x \in D = [a,b]$ satisfying the conditions

(*1) $$f_1(x) = 0, (\forall y \in D)\,(f_1(y) = 0, f_2(y) \geq 0 \Rightarrow g(x) \leq g(y))$$

Recall that the general definition of the feasible cell (i.e. Definition 4.2) is logically based on the fact (4.4) namely on the its part

(*2) $$\varphi(x_1,....) \Rightarrow m_r(\varphi)\,(C_r(x_1),...)$$

Accordingly for the second formula in (*1) we have the following argument

This formula is logically equivalent to

$(\forall y \in D)\,(f_1(y) > 0 \vee f_1(y) < 0 \vee f_2(y) < 0 \vee g(x) \leq g(y))$

and in view of (*2), i.e. by Definition 4.2. a cell $\Delta \in \mathcal{D}_r(D)$ is feasible iff it satisfies the condition

$$(\forall Y \in \mathcal{D}_r(D))(M(f_1)(Y) > 0 \vee m(f_1)(Y) < 0$$
$$\vee m(f_2)(Y) < 0 \vee m(g)(X) \leq M(g)(Y))$$

i.e. the condition

$$(\forall Y \in \mathcal{D}_r(D))(M(f_1)(Y) \leq 0, m(f_1)(Y) \geq 0, m(f_2)(Y) \geq 0$$
$$\Rightarrow m(g)(X) \leq M(g)(Y))$$

which is obviously useless; since from $M(f_1)(Y) \leq 0$, $m(f_1)(Y) \geq 0$ it follows that $f_1(y) = 0$ for all $y \in Y$.

However, about the second formula in (*1) we also have the following argument

Omitting the part $(\forall y \in D)$ from the formula we obtain the implication

(*3) $$f_1(y) = 0, f_2(y) \geq 0 \Rightarrow g(x) \leq g(y)$$

Suppose that $y_0 \in D$ is any solution of $f_1(y) = 0$. Putting $y = y_0$ from (*3) the implication

$$f_2(y_0) \geq 0 \Rightarrow g(x) \leq g(y_0)$$

follows, which is logically equivalent to

$$f_2(y_0) < 0 \vee g(x) \leq g(y_0)$$

Let X, Y_0 be elements of $\mathcal{D}_r(D)$ such that $x \in X, y_0 \in Y_0$. Then using only axiom (0.1) we obviously have the implications

$$f_2(y_0) < 0 \Rightarrow m(f_2)(Y_0) < 0; \quad g(x) \leq g(y_0) \Rightarrow m(g)(X) \leq M(g)(Y_0)$$

from which the following implication

$$(f_2(y_0) < 0 \vee g(x) \leq g(y_0)) \quad \Rightarrow \quad m(f_2)(Y_0) < 0 \vee m(g)(X) \leq M(g)(Y_0)$$

follows.

In fact this implication is an example of the general implication (*2). Now denote by $Sol_r(D)$ the set of all $Y \in \mathcal{D}_r(D)$ which have at least one solution of the equation $f_1(y) = 0$ with $y \in D$. Then in virtue of the proved implication we have this implication

$$(\forall y \in D)\,(f_1(y) = 0, f_2(y) \geq 0 \Rightarrow g(x) \leq g(y))$$
$$\Rightarrow (\forall Y \in Sol_r(D))\,(m(f_2)(Y) \geq 0 \Rightarrow m(g)(X) \leq M(g)(Y))$$

This implication provides a new idea how to define the feasibility criterium. Let us stop this argument and pass to the general case. About conditions (5.19) we put the following assumption (like (5.15))

(5.20) *If $k > s$ and at some point $(x_1, ..., x_n) \in D$ the formulas*

$$f_1(x_1, ..., x_n) = 0, \; ..., \; f_s(x_1, ..., x_n) = 0,$$
$$f_{s+1}(x_1, ..., x_n) \geq 0, \; ..., \; f_k(x_1, ..., x_n) \geq 0$$

are satisfied then in each neighbourhood $N(x_1, ..., x_n)$ of this point there is a point $(x_{1o}, ..., x_{no}) \in D$ satisfying the formulas

$$f_1(x_{1o}, ..., x_{no}) = 0, \; ..., \; f_s(x_{1o}, ..., x_{no}) = 0,$$
$$f_{s+1}(x_{1o}, ..., x_{no}) > 0, \; ..., \; f_k(x_{1o}, ..., x_{no}) > 0.$$

Using this assumption one can prove[9] the following equivalence (like (5.16))

(5.21)

$$(\forall\,(y_1, ..., y_n) \in D)\,[(f_1(y_1, ..., y_n) = 0, ... f_s(y_1, ..., y_n) = 0, f_{s+1}(y_1, ..., y_n) \geq 0, ...,$$
$$f_k(y_1, ..., y_n) \geq 0) \Rightarrow g(x_1, ..., x_n) \leq g(y_1, ..., y_n)]$$

is equivalent to

$$(\forall\,(y_1, ..., y_n) \in D)\,[(f_1(y_1, ..., y_n) = 0, ... f_s(y_1, ..., y_n) = 0, f_{s+1}(y_1, ..., y_n) > 0, ...,$$
$$f_k(y_1, ..., y_n) > 0) \Rightarrow g(x_1, ..., x_n) \leq g(y_1, ..., y_n)]$$

[9] See the proof of (5.16)

Now denote by $Sol_r(D)$ the set of all cells $X \in D_r(D)$ which have at least one solution of the system $f_1(x_1, ..., x_n) = 0, ..., f_s(x_1, ..., x_n) = 0$. Then we have the following definition

(5.22) *An element $X \in Sol_r(D)$ is feasible if it satisfies the following condition*

$$(\forall Y \in Sol_r(D))\,(m(f_{s+1})(Y) > 0, ..., m(f_k)(Y) > 0 \Rightarrow m(g)(X) \leq M(g)(Y))$$

The corresponding solving procedure is quite similar to (5.1). The set of all solutions is equal to $\bigcap_{r \in \mathbb{N}} F'_r$, which is not difficult to prove.

However, the main problem is how to determine $Sol_r(D)$. In other words how to find a condition $Cond(\Delta)$ such that the equivalence

A cell Δ has at least one solution of the system $f_1(x_1, ..., x_n) = 0, ..., f_s(x_1, ..., x_n) = 0$ $\Leftrightarrow$ $Cond(\Delta)$

is true. Roughly speaking, some "parts" of $Cond(\Delta)$ may be, for instance:

(j) Δ is is "small enough".
(jj) For each of the functions $f_i (i \leq i \leq s)$ there are two vertices V_1, V_2 of Δ such that $f_i(V_1) \cdot f_i(V_2) \leq 0$

Now we are going to give another idea about Problem 5.3, by which this problem can be solved approximatively. Namely, let $\varepsilon > 0$ be a given "small" real number. Replacing (5.19) by the following inequalities

(5.23)

$$f_1(x_1, ..., x_n) \geq -\varepsilon, f_1(x_1, ..., x_n) \leq \varepsilon, ..., f_s(x_1, ..., x_n) \geq -\varepsilon, f_s(x_1, ..., x_n) \leq \varepsilon$$
$$f_{s+1}(x_1, ..., x_n) \geq 0, ..., f_k(x_1, ..., x_n) \geq 0$$

from Problem 5.3 we obtain a new one, denoted by: Problem 5.3(ε). This Problem 5.3(ε) has two basic properties:

(i) It is a problem of Class 5.1 with a $\leq$-positive formula.
(ii) Depending on the magnitude of ε, the solutions of this problem are approximative solutions of the original Problem 5.3.

At the end we are interested in conditions like (5.18) which may be added to feasibility criterion. Suppose that each of the functions $f_1, ..., f_k, g$ satisfies a condition of type (5.10). First we consider the case $k = s$ when (5.19) reads.

(5.24)
$$f_1(x_1, ..., x_n) = 0, ..., f_s(x_1, ..., x_n) = 0$$

Let M be the following matrix

$$\begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \\ \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_s}{\partial x_1} & \frac{\partial f_s}{\partial x_2} & \cdots & \frac{\partial f_s}{\partial x_n} \end{bmatrix}$$

whose members are calculated at some point c belonging to the set[10] S. Denote by $M_1, M_2, \ldots, M_t$ all the minors of the matrix M which contain some elements of the first row of M and whose order is[11] $\min(n, s+1)$. Then:

At the point c *all the equalities*

(5.25) $$M_1 = 0, \ldots, M_t = 0$$

are satisfied.

This is a classical result, a generalization of (5.18).

We notice that in order to solve Problem[12] 5.3 we can use equalities (5.25) in several ways:

1^0 If we use the method of the feasibility criterium (5.22) we can for $X \in Sol_r(D)$ require the following conditions

$$m(M_i)(X) \le 0 \le M(M_i)(X) \quad (i = 1, \ldots, t)$$

too.

2^0 If we use the method by ε-approximation, when inequalities (5.23) are involved, then we can to them include the following inequalities

$$M_i \ge -\varepsilon, \quad M_i \le \varepsilon \quad (i = 1, \ldots, t)$$

3^0 Another way to solve the problem is to solve the system of equations

(5.24) plus (5.25)

and after that among the solutions to find points at which the function g attains the minimum value.

Finally we point out that equalities (5.25) are also satisfied in case of conditions (5.19) when $k > s$, but under the following assumption

At a point c$\in S$ are satisfied the conditions

$$f_1(c) = 0, \ldots, f_s(c) = 0, f_{s+1}(c) > 0, \ldots, f_k(c) > 0$$

Problem 5.4. *Now we consider the general problem of Class 5.1 allowing that in the formula φ the operators of type*

$$\min\nolimits_{x \in [a,b]}, \quad \max\nolimits_{x \in [a,b]}$$

may occur. For instance, such a φ is the formula:

(*) $$\min_{x \in [a,b]} \max_{y \in [c,d]} f(x,y,z) \le \max_{x \in [a,b]} g(x)$$

where f, g are given m-M functions.

[10] Recall, S is the set of all points at which the function g attains the minimum value, under the condition (5.19) i.e. (5.24).

[11] Of course, t is a constant, which can be easily calculated.

[12] Instead of (5.19) we have equalities (5.24)

However, the new Class 5.1 is not wider than the old one, for the following fact holds:

(5.26) *For any positive $<, \le$-formula φ containing the operators* min, max *one can effectively find a new positive $<, \le$-formula φ' such that φ' contains none of the symbols* min, max *and φ' is logically equivalent to φ.*

The proof is based on the following general facts:

Lemma 5.1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then:*

1^0 *For some $c, d \in [a,b]$ the equalities*

$$f(c) = \min_{x \in [a,b]} f(x), \quad f(d) = \max_{x \in [a,b]} f(x)$$

hold.

2^0 *For any real number p the following equivalences*

$$p \ge \min_{x \in [a,b]} f(x) \Leftrightarrow (\exists x \in [a,b])\, p \ge f(x)$$

$$p \le \min_{x \in [a,b]} f(x) \Leftrightarrow (\forall x \in [a,b])\, p \le f(x)$$

$$p \ge \max_{x \in [a,b]} f(x) \Leftrightarrow (\forall x \in [a,b])\, p \ge f(x)$$

$$p \le \max_{x \in [a,b]} f(x) \Leftrightarrow (\exists x \in [a,b])\, p \le f(x)$$

hold. Also in these equivalences one may replace the symnols $\ge, \le$ by $>, <$ respectively[13].

We shall demonstrate the idea of the proof of (5.26) by the following example.

Example 5.1. *Let $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be a given m-M function. Find*

(*) $$\min_{x_1 \in [a_1,b_1]} \max_{x_2 \in [a_2,b_2]} f(x_1, x_2).$$

Solution. In order to do this, we solve in $c_1 \in [a_1, b_1], c_2 \in [a_2, b_2]$ the following equation

(**) $$f(c_1, c_2) = \min_{x_1 \in [a_1,b_1]} \max_{x_2 \in [a_2,b_2]} f(x_1, x_2).$$

The elimination of the symbols min, max is as follows

$$f(c_1, c_2) = \min\nolimits_{x_1 \in [a_1,b_1]} \max\nolimits_{x_2 \in [a_2,b_2]} f(x_1, x_2)$$
$$\Leftrightarrow (\forall x_1 \in [a_1, b_1])\, f(c_1, c_2) \le \max\nolimits_{x_2 \in [a_2,b_2]} f(x_1, x_2)$$
$$\Leftrightarrow (\forall x_1 \in [a_1, b_1])\, (\exists x_2 \in [a_2, b_2])\, f(c_1, c_2) \le f(x_1, x_2)$$

[13] For instance, we have the following equivalence

$$p > \min_{x \in [a,b]} f(x) \Leftrightarrow (\exists x \in [a,b])\, p > f(x)$$

Consequently the related definitiion of feasible Cartesian products reads

$C_1 \times C_2$ is feasible

$$\Leftrightarrow \quad (\forall X_1 \in C_r[a_1, b_1]) \, (\exists X_2 \in C_r[a_2, b_2]) \, m(f) \, (C_1 \times C_2) \leq M(f) \, (X_1 \times X_2)$$

Remark 5.3. (due to E.Agović). *In order to find (*) we do not need to find all c_1, c_2 in (**). Having this in mind, for c_1, c_2 we impose the following additional condition*

$$f(c_1, c_2) = \max_{x_2 \in [a_2, b_2]} f(c_1, x_2)$$

Consequently to the feasibility criterium we add this request

$$(\forall X_2 \in C_r[a_2, b_2]) \, m(f) \, (C_1 \times X_2) \leq M(f) \, (C_1 \times C_2)$$

Remark 5.4. *In a way similar to that employed in Example 5.1 one can generally find*

$$(m_1 x_1) \, (m_2 x_2) \ldots (m_n x_n) \, f(x_1, \ldots, x_n)$$

where $f : D \to \mathbb{R}$ is a given m-M function and the symbol $(m_i x_i)$ stands optionally for

$$\min_{x_i \in [a_i, b_i]} \quad \text{or} \quad \min_{x_i \in [a_i, b_i]}$$

Problem 5.5. *This problem belongs to the* **Interval Mathematics.** *Namely, consider the general problem of Class 5.1 supposing that in the formula φ some constants $c_1, c_2, \ldots, c_k$ appear which we do not know exactly. Instead, we are given certain constants L_i, R_i such that $L_i \leq c_i \leq R_i$ $(i = 1, 2, \ldots, k)$. Accordingly, the formula φ will be also denoted by $\varphi(c_1, \ldots, c_k)$. A problem of Class 5.1 with such a φ will be called:*

5.1-problem wih boundaries $\quad L_i \leq c_i \leq R_i \quad (i = 1, \ldots, k)$

As we shall see any such problem can be translated to a genuine problem of Class 5.1 (see (5.27), (5.28) below). To prove this, let us first consider any 5.1.(ii)-problem. For instance, such a problem is stated in

Example 5.2. *Examine the truth of the formula*

$$(\forall x \in [1.4, 1.5]) \, x^2 \geq 1.8\ldots$$

where 1.8... is a constant satisfying the boundaries

$$1.8 \leq 1.8\ldots \leq 1.9$$

Solution. Obviously this problem is logically equivalent to the following problem of Class 5.1.(ii)

Is the formula $\quad (\forall c \in [1.8, 1.9]) \, (\forall x \in [1.4, 1.5]) \, x^2 \geq c$ *true or false*

Generally:

(5.27) *A problem:*

Is the formula $\varphi(c_1, \ldots, c_k)$ with boundaries $L_i \leq c_i \leq R_i$ true or false

is logically equivalent to the problem

Is the formula $(\forall c_1 \in [L_1, R_1]) \ldots (\forall c_k \in [L_k, R_k]) \, \varphi(c_1, \ldots, c_k)$ true or false.

Now we shall treat 5.1.(i)-problems with boundaries $L_i \leq c_i \leq R_i$ $(i = 1, \ldots, k)$. For instance, such a problem is encountered in

Example 5.3. *Find $x \in [1, 2]$ such that $x^2 = c$ where c is a constant with these information $1.69 \leq c \leq 1.96$ only.*

Solution. Obviously the best information on x is expressed by the inequalities $1.3 \leq x \leq 1.4$. This conclusion can be divided in the following two implications

The first reads:

(*) *For all $c \in [1.69, 1.96]$ the implication $x^2 = c, x \in [1, 2] \Rightarrow 1.3 \leq x \leq 1.4$ is true.*

The second reads:

(**) *If x, with $1.3 \leq x \leq 1.4$, is any number then for some $c \in [1.69, 1.96]$ the conditions $x^2 = c$; $x \in [1, 2]$ are true.*

Notice that about (*) we have the following reformulations

(*) $\Leftrightarrow (\forall c \in [1.69, 1.96]) \; (x^2 = c, x \in [1, 2] \;\Rightarrow\; 1.3 \leq x \leq 1.4)$

$\Leftrightarrow (\exists c \in [1.69, 1.96]) \, (x^2 = c, x \in [1, 2]) \;\Rightarrow\; 1.3 \leq x \leq 1.4$

(By applying the following logically valid formula

$$(\forall c) \, (\alpha(c) \Rightarrow \beta) \Leftrightarrow ((\exists c) \, \alpha(x) \Rightarrow \beta)$$

provided that c is not a free variable in β.)

On the other hand, about (**) we have

(**) $\Leftrightarrow (1.3 \leq x \leq 1.4 \Rightarrow (\exists c \in [1.69, 1.96]) \, x^2 = c, x \in [1, 2])$

Combining the obtained results we have the following equivalence

$$1.3 \leq x \leq 1.4 \Leftrightarrow (\exists c \in [1.69, 1.96]) \, x^2 = c, x \in [1, 2]$$

Consequently we have the following conclusion:

The problem stated in Example 5.3. is logically equivalent to the following problem:

Find $x \in [1, 2]$ such that the formula $(\exists c \in [1.69, 1.96]) \, x^2 = c$ is true.

The reasoning employed in this example can be transfered to any 5.1.(i)-problem with given boundaries $L_i \leq c_i \leq R_i$. Namely:

(5.28) *Any 5.1.(i)-problem with given boundaries $L_i \leq c_i \leq R_i$ $(i = 1, \ldots, k)$ is logically equivalent to the following problem (of Class 5.1.(i))*

Find all value of $x_i \in I(x_i)$ $(i = 1, ..., k)$ such that the formula

$$(\exists c_1 \in [L_1, R_1]) \dots (\exists c_k \in [L_k, R_k])\, \varphi(c_1, ..., c_k)$$

is satisfied.

Problem 5.6. *Let $A, B \subseteq \mathbb{R}$ be given sets and $\varphi(x, y)$ a positive $\leq$-formula whose free variables are x, y. Suppose that the formula*

(*) $\qquad (\forall x \in A)\, (\exists y \in B)\, \varphi(x, y)$

is true. How can we determine for y the best[14] *constants $c_1, c_2 \in \mathbb{R}$ such that the double inequality $c_1 \leq y \leq c_2$ holds?*

Solution. Obviously the implication

$$\varphi(x, y) \ \Rightarrow\ c_1 \leq y \leq c_2 \quad (x \text{ is any element of } A)$$

should be true. In other words we have the implication

$$(\forall x \in A)\, (\varphi(x, y) \ \Rightarrow\ c_1 \leq y \leq c_2)$$

which is logically equivalent to the following implication

$$(\exists x \in A)\, \varphi(x, y) \Rightarrow c_1 \leq y \leq c_2$$

The meaning of the last implication is

If for some $x \in A$ the formula $\varphi(x, y)$ is satisfied then the corresponding y must be between c_1, c_2.

According to this in order to solve Problem 5.6 proceed as follows

Take formula $(\exists x \in A)\, \varphi(x, y)$ and solve it for $y \in B$.

Applying procedure (5.1) we step-by-step obtain the set F_r' of all feasible r-cells, by which we can approximatively determine the constants c_1, c_2.

Example 5.4. *Find $z \in [-7, 25]$ such that the condition*

$$(\forall x \in [0, 4])\, (\exists y \in [3, 5])\, y^2 - x^2 = z$$

is satisfied.

Solution. The ordinary binary trees are used. The calculations are carried out only up to step 7. The member of feasible cells is 2 in each step, and z satisfies: $8.75 \leq z \leq 9.25$.

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[14]That means that inequality $c_1 \leq y \leq c_2$ is implyied by (*), but also for some $x_1, x_2 \in A$ we have respectively $\varphi(x_1, c_1), \varphi(x_2, c_2)$.

6. FINDING FUNCTIONS AS SOLUTIONS OF A GIVEN m-M CONDITION

In this section we state how approximatively to determine functions satisfying a given m-M condition[1], which is some functional condition, or some difference condition, or differential equation.

1. Let $\varphi(x, y)$ be a positive $\leq$-formula, quantifier-free, and whose free variables are x, y. Replacing y by a term $f(x)$, where f is a function symbol from the formula $\varphi(x, y)$ we obtain

(6.1) $\qquad \varphi(x, f(x))$

which we shall call "an m-M (functional) condition". Let $A, B \subseteq \mathbb{R}$ be given segments and $f : A \to B$ a function satisfying the condition $(\forall x \in A)\, \varphi(x, f(x))$. Then we say that *$f$ is a solution of condition* (6.1).

In the sequel we are going to describe a procedure by which one can step-by-step approximatively determine all such functions (if any exists). We shall use the following denotations

* *X will be a sequence of some subsegments (i.e. cells) of the segment A. By $l(X)$ is denoted the number of its elements.*

* *If $P \in X$ then by $F(P)$, $F_1(P)$ will be denoted some sequences of subsegments of the segment B; $l(F(P)), l(F_1(P))$ are the numbers of their elements.*

We also use the following convention:

Two segments of the forms $[p, q]$, $[r, s]$ are called **neighbouring** *if $q = r$ or $s = p$.*

In fact in the procedure we search certain solution $x \in A, y \in B$ of $\varphi(x, y)$, having in mind that y should be a function of x. The procedure reads:

(6.2)(i) *If $m(\varphi)\,(A \times B)$ is false the procedure stops and the result is:*

(6.1) has none f-solution.

In the opposite case we take:

$l(X) = 1, X_1 = A, l(F(A)) = 1, B$ is the unique element of $F(A)$;

and go to (ii).

(ii) In turn we take $P = X_i$ $(1 \leq i \leq l(X))$ and for each of them we do the following:

From the sequence $F(P)$ we form a new sequence $F_1(P)$ consisting of all elements $Q \in F(P)$ for which the condition $m(\varphi)(P \times Q)$ holds. If the sequence $F_1(P)$ is empty then the procedure stops and the result is:

---

[1]Throughout the section 'm-M condition' is a generalization of the notion 'm-M equation'.

(6.1) has none f-solution.

In the opposite case we first make unions of all neighbouring elements of the sequence $F_1(P)$ and in such a way we obtain a new sequence, which we call $F_1(P)$ again[2]

After $P = X_{l(X)}$ is being processed we go to (iii).

(iii) In this step we have already determined the desire function f approximatively:

Namely, for any $x \in A$ let $P \in X$ be a segment containing this x. Then $f(x)$ may be any number which is any element of some element of $F_1(P)$.

If we want to continue the procedure then we do the following:

First, for each $P = X_i$ $(1 \le i \le l(X))$ we do the following:

We decompose P into smaller subsegments, say $P_1, ..., P_r$ and temporarily extend the function F_1 by the conditions

$$F_1(P_1) = ... = F_1(P_r) = F_1(P)$$

Let X_1 be the sequence of all such subsegments for all elements $P \in X$. Next, in turn to each element $P \in X_1$ we consider the related sequence $F_1(P)$ and decompose all its elements into some smaller subsegments. In such a way from $F_1(P)$ we obtain a new sequence named $F(P)$. We put $X = X_1$ and go to (ii).

In a similar way one can approximatively solve any functional condition like (6.1) under this restriction:

All unknown functions have the same number of arguments.

For instance, the functional conditions

$$\varphi(x, f(x), g(x)), \quad \psi(x, y, h(x,y), k(x,y), m(x,y))$$

$(f, g, h, k, m$ are unknown functions)

belong to this class. However, the functional condition

(6.3) $$\varphi(x, f(x), y, g(x,y))$$

obviously is not a member of the class. Solving procedure for such conditions in some details differs from procedure (6.2). For example to solve (6.3) we proceed as follows:

We replace (6.3) by the following functional condition

$$\varphi(x, f_1(x,y), y, g(x,y))$$

with two unknown functions[3] *f_1, g. Then using a procedure like (6.2) we seek those of its solutions which are solutions of (6.3) too. Namely, suppose that in some step for x and y we have all together the following subsegments*

$$X_1, ..., X_p, Y_1, ..., Y_q$$

respectively. Let $P = X_i$ be any of these $X_1, ..., X_p$. To define the sequence[4] *$F(P)$ we consider all sequences*

(*) $$F_1(P, Y_1), ..., F_1(P, Y_q)$$

and then: any subsegment $[a, b]$ is an element of $F(P)$ if and only if[5] *this subsegment belongs to each member of the sequences (*). If $F(P)$ is empty*[6] *sequence the procedure stops with the conclusion that (6.3) has no solutions.*

2. Let now $\varphi(x, y, z)$ be a positive $\le$-formula, quantifier-free, whose all free variables are x, y, z. Replacing y, z by the following terms f_x, f_{x+h} respectively from the formula $\varphi(x, y, z)$ we obtain

(6.4) $$\varphi(x, f_x, f_{x+h}) \qquad (h \text{ is a given positive constant})$$

which we are going to call "an m-M difference condition". Concerning (6.4) the main problem is:

(6.5) *Let a, a', b (with $a < b$) be some given real numbers. Giving to x and f_x initial values a, a' respectively determine a finite sequences (if any exists)*

$$f_a, f_{a+h}, ..., f_{a+nh} \qquad (a + nh \text{ is } b)$$

such that (6.4) is satisfid for every $x \in \{a, a+h, ..., a+(n-1)h\}$.

In the sequel we shall describe a procedure by which one can approximatively determine all such sequences[7], under the restriction that $f_a, f_{a+h}, ..., f_{a+nh}$ belong to a given segment $B \subset \mathbb{R}$.

As a matter of fact problem (6.5) is logically equivalent to the following:

(6.6) *Solve for $f_{a+h}, ..., f_{a+nh} \in B$ the system*

$$\begin{aligned} &\varphi(a, a', f_{a+h}), \\ &\varphi(a+h, f_{a+h}, f_{a+2h}), \\ &\vdots \\ &\varphi(a+(n-1)h, f_{a+(n-1)h}, f_{a+nh}) \end{aligned}$$

[2] For instance, if $F_1(P)$ is the sequence $[1,2], [7,8], [2,3], [6,7], [3,4], [9,10]$ then the new $F_1(P)$ is the following sequence $[1,4], [6,8], [9,10]$

[3] Both of f_1 and g are functions of 2 arguments.

[4] Of course, its elements are some subsegments which approximatively determine the function f.

[5] This is a way to express that $f_1(x,y)$ should not be a function of y.

[6] i.e. with no members.

[7] if any exists

The procedure reads:

(6.7) *Using a procedure of type (5.1) we approximatively solve for*[8] f_{a+h} *the first formula, i.e. the formula* $\varphi(a, a', f_{a+h})$. *In such a way we obtain as the result some set* F_{a+h}*-union of some subsegments*[9] *of the set* B. *Next, we go to the second formula* $\varphi(a+h, f_{a+h}, f_{a+2h})$ *which we shall solve for* f_{a+2h} *under the assumption* $f_{a+h} \in F_{a+h}$. *In other words*[10], *we need to solve for* $f_{a+2h} \in B$ *this formula*

$$(\exists y \in F_{a+h})\, \varphi(a+h, y, f_{a+2h})$$

Again we apply a corresponding procedure of type[11] *(5.1) and for* f_{a+2h} *we determine some set* f_{a+2h}*-union of some subsegments*[12] *of the set* B. *Similarly we proceed with the remaining formulas* $\varphi(a+2h, f_{a+2h}, f_{a+3h})$, ..., $\varphi(a+(n-1)h, f_{a+(n-1)h}, f_{a+nh})$. *So, solving the formula*

$$(\exists y \in F_{a+2h})\, \varphi(a+2h, y, f_{a+3h})$$

we obtain the set F_{a+3h}; *solving the formula*

$$(\exists y \in F_{a+3h})\, \varphi(a+3h, y, f_{a+4h})$$

we obtain the set F_{a+4h}, *and so on. Finally, the desired sequence is approximatively determined by these conclusions:*

$$f_a = a', f_{a+h} \in F_{a+h}, ..., f_{a+nh} \in F_{a+nh}$$

Of course, it can happen that for some i *the set* F_i *is empty, when the procedure should be stopped with the conclusion that the desired sequence does not exist.*

We point that besides (6.4) one can in a similar way solve various other difference conditions like $\varphi(x, f_x, f_{x+h}, f_{x+2h})$, and so on.

3. In this part we state a procedure by which one can approximatively solve a given differential equation. Let

$$E(x, f(x), f'(x)) = 0 \tag{6.8}$$

be a given differential equation having a solution $f : A \to B$. Denote by C a set with the property

$$f'(x) \in C \qquad \text{whenever } x \in A$$

Suppose that there exists $f''(x)$ for $x \in A$, and that the function $E(x, y, z)$ (with $x \in A, y \in B, z \in C$) is differentiable. Additionally suppose

[8] This means that we use the solving procedure up to some step r, and this r is of our choice.

[9] In fact these subsegments are corresponding feasible cells.

[10] See Problem 5.5, Example 5.3.

[11] up to some step r, r is a number of our choice.

[12] In fact, they are corresponding feasible cells.

(6.9) *There are positive constants* K_1, K_2 *such that*

$$\left|\frac{\partial E}{\partial z}\right| \le K_1, \quad |f''(x)| \le K_2$$

for every $x \in A, y \in B, z \in C$.

Then one can immediately prove the following assertion

If x *and* $x+h$ *(with* $h > 0$*) are any elements of* A *then the inequality*

$$\left|E\left(x, f(x), \frac{f(x+h) - f(x)}{h}\right)\right| \le K_1 K_2 h \tag{6.10}$$

holds.

From inequality (6.10) one can easily get an idea for solving equation (6.8). Namely, first suppose that $E(x, y, z)$ is an m-M function. Then to (6.10) one can assign the corresponding m-M difference condition, say expressed in this manner[13]

$$\left|E\left(x, f_x, \frac{1}{h}(f_{x+h} - f_x)\right)\right| \le K_1 K_2 h \tag{6.11}$$

In such a way we obtain an example of the difference conditions of type (6.4). Consequently we can apply a procedure of type[14] (6.7). Of course, in order to do this we have to know the constants K_1, K_2 in advance. About $K1$ it suffices to suppose that $\frac{\partial E}{\partial z}$ is an m-M function. To find K_2, briefly said, one can besides equation (6.8) employ the equation

$$\frac{\partial E}{\partial x} + \frac{\partial E}{\partial y} f'_y + \frac{\partial E}{\partial z} f''_z = 0$$

At the end we emphasize that by solving difference condition (6.11) we in fact approximatively determine all solutions of differential equation (6.8) with initial condition $f(a) = a'$. But, if solving procedure stops then we conclude that equation (6.8) has none such solution.

[13] Now, f is used as a sequence-symbol.

[14] The members a, b are determined by the set A, while a, h (with $a' \in B, h > 0$) are of our choice.

7. APPENDIX

This part contains a new much simpler proof of Theorem 4.3

Let $\phi(x_1,...,x_m)$ be a $<$ −positive formula for which $x_1,...,x_m$ are all its free variables, and $y_1,...,y_k$ all its bounded variables. Suppose that to each of the segments $I(x_i)$, $I(y_j)$ one cell-decomposition is assigned (see Definition 1.2). Our aim is to simplify the formula ϕ in the following sense: *to find a term t such that ϕ becomes equivalent to the inequality $t > 0$ and also that ϕ and $t > 0$ have equivalent m-M pairs* with respect to Definition 4.1.

First, let $\phi'(x_1,...,x_m)$ be a prenex form of $\phi(x_1,...,x_m)$, i.e. a formula equivalent to ϕ and having the following form

(7.1) $(q_1 y_1 \in I(y_1)) \ ... \ (q_k y_k \in I(y_k)) \quad \psi(y_1,...,y_k,x_1,...,x_m)$

where each of q_j is one of the quantifiers and ψ is quantifier-free. For instance, if ϕ is the formula

$$[(\forall y_1 \in I(y_1)) \ f_1(x_1,y_1) < f_2(y_1) \ \wedge \ f_2(x_1) < 1) \quad \vee \ [f_1(x_1,x_2) < f_3(x_1,x_2)] \ \wedge \ (\exists y_2 \in I(y_2)) f_3(y_2,x_2) < f_2(x_1)$$

then a prenex form of it is

$$(\forall y_1 \in I(y_1)) \ (\exists y_2 \in I(y_2)) \ [f_1(x_1,y_1) < f_2(y_1) \ \wedge \ f_2(x_1) < 1) \ \vee \ [f_1(x_1,x_2) < f_3(x_1,x_2)] \ \wedge f_3(y_2,x_2) < f_2(x_1)$$

Bearing in mind Definition 4.1 it is easily seen that for every $r = 0, 1, ...$ the following equivalences

$$(Eq(\phi,\phi')) \qquad \begin{array}{l} M_r(\phi)(C_r(x_1),...,C_r(x_m)) \longleftrightarrow M_r(\phi')(C_r(x_1),...,C_r(x_m)) \\ m_r(\phi)(C_r(x_1),...,C_r(x_m)) \longleftrightarrow m_r(\phi')(C_r(x_1),...,C_r(x_m)) \end{array}$$

hold. In the next step using the functions $max(x,y), min(x,y)$ we shall eliminate the logical connectives $\wedge$, $\vee$. Namely, by means of the following equivalences

$$\begin{array}{c} A < B \longleftrightarrow B - A > 0 \\ A > 0 \wedge B > 0 \longleftrightarrow min(A,B) > 0 \\ A > 0 \vee B > 0 \longleftrightarrow max(A,B) > 0 \end{array}$$

the formula ψ in (7.1) can be transformed to a formula of the form

$$g(y_1,...,y_k,x_1,...,x_m) > 0$$

where the term g is built up by using min and max. In such a way from the formula (7.1) we obtain the following equivalent formula

(7.2) $(q_1 y_1 \in I(y_1)) \ ... \ (q_k y_k \in I(y_k)) \ g(y_1,...,y_k,x_1,...,x_m) > 0$

briefly denoted by $\phi''(x_1,...,x_m)$. Now again bearing in mind Definition 4.1 is easily seen that the equivalences of the form $(Eq(\phi, \phi'))$, i. e. the equivalences $(Eq(\phi', \phi''))$ are satisfyied. Next, using the operators of the form $min_{x \in I(x)}$,

$max_{x \in I(x)}$ we are going to eliminate the quantifiers. Namely, we assign to the formula (7.2) the following formula ϕ'''

(*1) $\mu(q_1)_{y_1 \in I(y_1)} \cdots \mu(q_k)_{y_k \in I(y_k)} \ g(y_1,...,y_k,x_1,...,x_m) > 0$

where the mapping μ is defined by: $\mu(\forall) = min$, $\mu(\exists) = max$. For instance, to the formula $(\forall y_1 \in I(y_1))(\exists y_2 \in I(y_2)) \ g(y_1,y_2,x_1) > 0$ the corresponding formula reads $min_{y_1 \in I(y_1)} max_{y_2 \in I(y_2)} \ g(y_1,y_2,x_1) > 0$.

Between (7.2) and (*1), i.e. ϕ'' and ϕ''' there is the following equivalence

$$(7.3) \quad (q_1 y_1 \in I(y_1)) \ ... \ (q_k y_k \in I(y_k)) \ g(y_1,...,y_k,x_1,...,x_m) > 0 \\ \longleftrightarrow \mu(q_1)_{y_1 \in I(y_1)} \cdots \mu(q_k)_{y_k \in I(y_k)} \ g(y_1,...,y_k,x_1,...,x_m) > 0$$

For instance, concerning the given example above we have the following equivalence

$$(*2) \quad (\forall y_1 \in I(y_1))(\exists y_2 \in I(y_2)) \ g(y_1,y_2,x_1) > 0 \\ \longleftrightarrow min_{y_1 \in I(y_1)} max_{y_2 \in I(y_2)} \ g(y_1,y_2,x_1) > 0$$

Equivalence (7.3) follows immediately from the following general equivalences

$$(*3) \quad (i) \ (\forall x \in [a,b]) \ f(x) > 0 \longleftrightarrow min_{x \in [a,b]} \ f(x) > 0 \\ (ii) \ (\exists x \in [a,b]) \ f(x) > 0 \longleftrightarrow max_{x \in [a,b]} \ f(x) > 0$$

where a, b are any reals and $f : [a,b] \to R$ a continuous function. For instance, such a proof for (*2) reads

$$min_{y_1 \in I(y_1)} max_{y_2 \in I(y_2)} \ g(y_1,y_2,x_1) > 0$$

$$\longleftrightarrow (\forall y_1 \in I(y_1)) max_{y_2 \in I(y_2)} \ g(y_1,y_2,x_1) > 0$$

$$\longleftrightarrow (\forall y_1 \in I(y_1))(\exists y_2 \in I(y_2)) \ g(y_1,y_2,x_1) > 0$$

since $max_{y_2 \in I(y_2)} \ g(y_1,y_2,x_1) > 0 \longleftrightarrow (\exists y_2 \in I(y_2)) \ g(y_1,y_2,x_1) > 0$

In the next step we shall define an m-M pair for the formula ϕ'''. This formula has the form $t > 0$. For this t we define an m-M pair by the following equalities

$$(*4) \quad m(t)(C(x_1) \times \ ... \ \times C(x_m)) \\ = \mu(q_1)_{Y_1 \in \mathcal{D}_r(I(y_1))} \cdots \mu(q_k)_{Y_k \in \mathcal{D}_r(I(y_k))} \ m(g)(Y_1 \times ... \times Y_k \times C(x_1) \times ... \times C(x_m))$$

$$(*5) \quad M(t)(C(x_1) \times \ ... \ \times C(x_m)) \\ = \mu(q_1)_{Y_1 \in \mathcal{D}_r(I(y_1))} \cdots \mu(q_k)_{Y_k \in \mathcal{D}_r(I(y_k))} \ M(g)(Y_1 \times ... \times Y_k \times C(x_1) \times ... \times C(x_m))$$

It is not difficult to prove the correctness of these definitions using induction on the number k. We will explain two basic steps in case $k = 1$ concerning the formula (*4). Namely, in that case we should prove the following inequalities

(*6) $min_{y_1 \in I(y_1)} g(y_1,x_1,...,x_m) \geq min_{Y_1 \in \mathcal{D}_r(I(y_1))} m(g)(Y_1 \times C(x_1) \times ... \times C(x_m))$

(*7) $max_{y_1 \in I(y_1)} g(y_1,x_1,...,x_m) \geq max_{Y_1 \in \mathcal{D}_r(I(y_1))} m(g)(Y_1 \times C(x_1) \times ... \times C(x_m))$

One proof of (*6) reads. Suppose that r and $x_1,...,x_m$ have some fixed value. Then $g(y_1,x_1,...,x_m)$ as a function of y_1 attains its minimum at some point $y_1' \in I(y_1)$. Thus, we have the following argument

$$
\begin{aligned}
&min_{y_1 \in I(y_1)} g(y_1, x_1, ..., x_m)\\
&\quad = g(y_1', x_1, ..., x_m)\\
&\quad \geq m(g)(C(y_1') \times C(x_1) \times ... \times C(x_m))\\
&\quad \geq min_{Y_1 \in \mathcal{D}_r(I(y_1))} m(g)(Y_1 \times C(x_1) \times ... \times C(x_m))
\end{aligned}
$$

One proof of (*7) reads. Suppose that r and $x_1, ..., x_m$ have some fixed value. Then $m(g)(Y_1 \times C(x_1) \times ... \times C(x_m))$ for some $Y_1' \in \mathcal{D}_r(I(y_1))$ attains the maximum value[2] Thus, we have the following argument

$$
\begin{aligned}
&max_{Y_1 \in \mathcal{D}_r(I(y_1))} m(g)(Y_1 \times C(x_1) \times ... \times C(x_m))\\
&\quad = m(g)(Y_1' \times C(x_1) \times .... \times C(x_m))\\
&\quad \leq g(y_1', x_1, ..., x_m) \qquad \text{(for some } y_1' \in I(y_1))\\
&\quad \leq max_{y_1 \in I(y_1)} g(y_1, x_1, ..., x_m)
\end{aligned}
$$

Bearing in mind (*4), (*5) and Definition 4.1 we see that $m_r(\phi''')(C(x_1),...,C(x_m))$, $M_r(\phi''')(C(x_1),...,C(x_m))$ are $M(t)(C(x_1) \times ... \times C(x_m))>0$, $m(t)(C(x_1) \times ... \times C(x_m))>0$ respectively.

Next, we point out that we again have an equivalence of type $Eq(\phi, \phi')$, namely the equivalence $Eq(\phi'', \phi''')$. This fact follows from the following general equivalences on reals $(\forall i \in \{1,n\}) a_i > 0 \longleftrightarrow min_{i \in \{1,n\}} a_i > 0$, and $(\exists i \in \{1,n\}) a_i > 0 \longleftrightarrow max_{i \in \{1,n\}} a_i > 0$, which together can be written in this way

(*8) $(qi \in \{1, n\}) a_i > 0 \longleftrightarrow \mu(q)_{i \in \{1,n\}} a_i > 0, \qquad \text{where } q \in \{\forall, \exists\}$

Hence, for $M_r(\phi''')(C(x_1), ..., C(x_m))$ we have the following equivalence chain

$$
\begin{aligned}
&M_r(\phi''')(C(x_1), ..., C(x_m))\\
&\longleftrightarrow m(t)(C(x_1) \times \; ... \; \times C(x_m)) > 0\\
&\longleftrightarrow \mu(q_1)_{Y_1 \in \mathcal{D}_r(I(y_1))} \cdots \mu(q_k)_{Y_k \in \mathcal{D}_r(I(y_k))}\, m(g)(Y_1 \times ... \times Y_k \times C(x_1) \times ... \times C(x_m)) > 0\\
&\longleftrightarrow (q_1 Y_1 \in \mathcal{D}_r(I(y_1))) \; ... \; (q_k Y_k \in \mathcal{D}_r(I(y_k)))\, m(g)(Y_1 \times ... \times Y_k \times C(x_1) \times ... \times C(x_m)) > 0\\
&\qquad\qquad \text{(by (*8))}
\end{aligned}
$$

According to (7.2) and Definition 4.1 we obtained $M_r(\phi'')(C(x_1), ..., C(x_m))$. In a similar way one can prove the equivalence $m_r(\phi''')(...) \longleftrightarrow m_r(\phi'')(...)$ too.

Now we shall prove Theorem 4.3. It suffices to prove part $\Rightarrow$ –part of (j) only. So, let be $\phi(x_1, ..., x_m)$. We have proved the equivalences $\phi \longleftrightarrow \phi' \longleftrightarrow \phi'' \longleftrightarrow \phi'''$ so, we conclude that $\phi'''(x_1, ..., x_m)$ holds. Further,ϕ''' has the form $t(x_1, ..., x_m) > 0$ which implies that $t(x_1, ..., x_m)$ is positive. By axioms (0.1),
(0.2) we conclude that there exists $r \in N$ such that $m(t)(C_r(x_1) \times ... \times C_r(x_m))>0$ also holds. Thies yields that $M_r(\phi''')(C(x_1), ..., C(x_m))$ holds. Since, we have the equivalences $M_r(\phi''')(...) \longleftrightarrow M_r(\phi'') \longleftrightarrow M_r(\phi')(...) \longleftrightarrow M_r(\phi)(..)$ we finally conclude that $M_r(\phi)(C(x_1), ..., C(x_m))$ holds and the proof is complete.

[2] The set $\mathcal{D}_r(I(y_1))$ is finite

REFERENCES

[1] Alfred, G. and Herberger, J., *Introduction to Interval Computations*, Academic Press, A subsidiary of Harcourt Brace Jovanovich, Publishers New York, London, Paris, San Diego, San Francisco, Sao Paolo, Sydney, Tokyo, Toronto, 1983

[2] Collatz, L., *Functional Analysis and Numerische Mathematik*, Springer-Verlag, Berlin-Göttingen-Heidelberg.

[3] Fletcher, R., *Practical Methods of Optimization, Vol.1, Unconstrained Optimization*, John Willey & Sons; Chichester, New York, Brisbone, Toronto; 1980.

[4] Fletcher, R., *Practical Methods of Optimization, Vol.2, Constrained Optimization*, John Willey & Sons; Chichester, New York, Brisbone, Toronto; 1980.

[5] Hagemon, L.A. and Joung, D.M., *Applied Iterative Methods*, Academic Press; New York, London, Toronto, Sydney, San Francisko, 1981.

[6] Mendelson, E., *Introduction to Mathematical Logic*, New York, 1964.

[7] Prešić, S.B. *m-M Calculus*, Matematički Institut, Beograd, 1996.

Izdavač: Matematički institut, Beograd, Kneza Mihaila 35

Recenzenti: Gradimir Milovanović (Niš)
Miodrag Rašković (Kragujevac, Beograd)

Primljeno za štampu 26. juna 1996.
odlukom Naučnog veća Matematičkog instituta

Tehnički urednik: Dragan Blagojević

Kompjuterski slog: Dragan Urošević

Štampa: "Galeb", Zemun, Kej oslobođenja 73

Štampanje završeno oktobra 1997.

Klasifikacija Američkog matematičkog društva
(AMS Mathematics Subject Classification): 65-02, 03B99

Univerzalna decimalna klasifikacija: 519.61, 510.63

CIP - Katalogizacija u publikaciji
Narodna biblioteka Srbije, Beograd

519.61

PREŠIĆ, Slaviša B.
m-M Calculus / Slaviša B. Prešić.
– (2nd revised ed.).
– Beograd : Matematički institut, 1997 (Zemun : Galeb).
– 87 str. : graf. prikazi ; 24 cm.

Tiraž 300. – Bibliografija: str. 87.
ISBN 86-80593-29-X

510.63
a) Numerička analiza b) Matematička logika

ID=57807884